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PREFACE

The present volume is the fourth in the series entitled Contributions to the Calculus of Variations, sponsored by the Department of Mathematics of the University of Chicago. It contains the theses on topics in the calculus of variations accepted for the Ph. D. degree at the University of Chicago during the four years 1938 to 1941, and the bibliography described below. All of the theses here presented were written under the direction of one or another of the four editors whose names are signed to this preface. They are listed in the order in which they were approved.

The preceding three volumes of the series contain theses, and a few other papers, submitted in 1930, in 1931-1932, and in 1933-1937, respectively. In the preface of the third volume it was incorrectly stated that the period included in that volume ended with June 1937. The thesis of Smiley, which appears there, was added at the last moment and is dated August 1937. It was the last one in the domain of the calculus of variations presented in that year, so that the third volume contains all of the theses for the years 1933 to 1937, inclusive.

For some years past the editors of this volume have cooperated with each other and with visitors from other universities in giving advanced courses entitled "Topics in the Calculus of Variations." One of the principal topics discussed has been the theory of the calculus of variations for multiple integrals. In connection with the studies made in that field a bibliography has been accumulated which has been exceedingly useful. With the hope that it may be equally valuable to others this bibliography is presented here as the last section in this book. The titles of memoirs up to 1930 were mostly contributed by Dr. Max Coral, and those for succeeding years by Dr. Herman H. Goldstine. Others also have assisted, and we are indebted to Mr. J. Ernest Wilkins, Jr. for some recent titles.

G. A. Bliss
L. M. Graves
M. R. Hestenes
W. T. Reid

THE UNIVERSITY OF CHICAGO
November 1941

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THE MINIMUM OF A FUNCTION OF INTEGRALS
IN THE
CALCULUS OF VARIATIONS

BY

CARROLL PARKER BRADY

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THE MINIMUM OF A FUNCTION OF INTEGRALS IN THE CALCULUS OF VARIATIONS

Introduction. In the fourth chapter of his "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes" Euler discusses a number of calculus of variations problems in which a function of several integrals is to be minimized or maximized [I, chap. 4].[1] For a general problem of that type [I, p. 161] he shows that the differential equations necessarily satisfied by a minimizing arc are the same as the equations that must be satisfied by an extremal arc for the isoperimetric problem of minimizing a single one of the integrals when the remaining integrals have prescribed values. Bolza [II] considered two of the special problems that Euler had discussed [I, pp. 148 and 153], and by means of the theories of associated isoperimetric problems he established the existence of arcs for which the functions of integrals considered took on strong relative extreme values. No doubt the method used by Bolza could be generalized so as to apply to the more general problem to be considered in this paper. The problems treated by Euler and Bolza may also be transformed into problems of Bolza in the calculus of variations. It is the purpose of this paper to apply the theory of the problem of Bolza to a general problem in which the functional to be minimized is a function of integrals. By so doing we develop a theory directly applicable to problems of the type discussed by Bolza and Euler.

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[1] The Roman numerals in square brackets refer to the bibliography at the end of this paper.

In Section 1 we formulate the problem to be considered in this paper, which we call problem A, and transform it into a problem B which is a problem of Bolza but not of the most general type. The method of the following pages is to interpret for problem A via the problem B the results already in the literature concerning the general problem of Bolza which is designated here as problem C. Section 2 below contains a discussion of the multiplier rule and of the extremals. We then consider, in Section 3, the conditions of Weierstrass and Clebsch. Preparatory to the formulation of two Mayer conditions in Section 5 we discuss the second variation and the accessory extremals in Section 4. An application of the sufficiency theorems for the problem of Bolza is made in Section 6. This procedure yields only a restricted strong relative minimum for problem A because the definitions of a strong relative minimum for problems A and B are not equivalent. A satisfactory theorem concerning strong relative minima for problem A can be deduced, however, by supplementing the results for problem B, as is done in Section 7, by methods similar to those used by Hestenes for isoperimetric problems. Finally, in Sections 8 and 9 we apply the more general theory here developed to the two special problems considered by both Euler and Bolza.

1. Formulation of the problem. As a generalization of the problems mentioned in the introduction and considered by Bolza we propose the following one. Consider a class $\mathfrak{M}$ of so-called admissible arcs in the xy-plane defined by functions of the form

$$y(x) \qquad (x_1 \leqq x \leqq x_2). \tag{1:0}$$

Let us suppose that for each of these arcs the integrals

$$I_\beta = \int_{x_1}^{x_2} h_\beta(x, y, y')dx \qquad (\beta = 1, \ldots, n) \tag{1:1}$$

and the function $g(I_1, \ldots, I_n)$ have well defined values. Then the problem to be considered is that of finding in the sub-class of admissible arcs which join two fixed points in the xy-plane one which minimizes the function g. In the following pages this problem will be referred to as problem A.

In order to provide an analytic basis for the study of this problem we consider an arc E belonging to the above mentioned sub-class of $\mathfrak{M}$ in which a minimizing arc is to be sought. We suppose that E is continuous and consists of a finite number of sub-arcs on each of which it has a continuously turning tangent. We assume that the functions $h_\beta(x, y, y')$ $(\beta = 1, \ldots, n)$ have continuous partial derivatives of at least the third order in a neighborhood R of the points (x, y, y') on E, that g has continuous partial derivatives of at least the second order in a neighborhood of the values $(I_1, \ldots, I_n)$ belonging to E, and finally that at that point $(I_1, \ldots, I_n)$ not all of the first derivatives of g vanish. An arc defined by functions of the form (1:0) will be said to be admissible if it is continuous and consists of a finite number of sub-arcs each of which has a continuously turning tangent, and if furthermore its elements (x, y, y') all lie interior to the region R. Such an arc will certainly give well defined values to the integrals I_β, and also to the function g provided that the region R is sufficiently small.

In the following pages we treat the foregoing problem A as a special case of the problem of Bolza in the calculus of variations. In order to do this we consider a class of arcs defined in an $(n + 2)$-dimensional space by functions of the form

$$(1{:}2) \qquad y_i(x) \qquad (x_1 \leqq x \leqq x_2;\ i = 0, 1, \ldots, n).$$

Such an arc will be termed admissible if it is continuous and consists of a finite number of sub-arcs having continuously turning tangents, and if its xy_0-projection, defined by the function $y_0(x)$, is admissible for the problem A described above. We designate the class of admissible arcs (1:2) by $\mathfrak{M}'$. The problem of Bolza corresponding to the problem formulated in the first paragraph is then that of finding in the sub-class of the class $\mathfrak{M}'$ satisfying the differential equations and end conditions

$$(1:3) \qquad h_\beta(x, y_0, y_0') - y_\beta' = 0 \qquad (\beta = 1, \ldots, n),$$

$$(1:4) \quad x_s - a_s = y_0(x_s) - y_{0s} = y_\beta(x_1) = 0 \ (s = 1, 2;\ \beta = 1, \ldots, n)$$

one which minimizes the function

$$(1:5) \qquad g[y_1(x_2), \ldots, y_n(x_2)].$$

In the following pages this problem will be designated as problem B.

We note that the statements of problems A and B here given are entirely equivalent. This follows because there is a one-to-one correspondence between the arcs of the sub-class of $\mathfrak{M}$ through the two given points 1 and 2 and the arcs of the sub-class of $\mathfrak{M}'$ for which the conditions (1:3) and (1:4) are satisfied, and since furthermore on corresponding arcs the values of the function g are the same. We are justified therefore in applying to the more special problem A formulated above some of the well known results from the theory of the problem of Bolza.

In order to facilitate the statement of the results from the theory of the problem of Bolza that we desire to use, we here state that problem as it appears in the literature [VIII, pp. 9-11]. In the following pages it will be designated as problem C. The problem will be formulated in terms of the notation used in the less general problem B just described. In stating problem B we were careful to make definitions and hypotheses that would ensure its appearance as an instance of the general problem of Bolza. We do not, therefore, give the similar though more extensive sets of definitions and hypotheses necessary to place the general theory on an analytic basis. With this understanding we then state the problem C of Bolza as that of finding in a class of admissible arcs defined in an $(n + 2)$-space of points $(x, y_0, y_1, \ldots, y_n)$ by functions of the form

$$y_i(x) \qquad (x_1 \leqq x \leqq x_2;\ i = 0, 1, \ldots, n) \tag{1:6}$$

and satisfying the differential equations and end conditions

$$\phi_\beta(x, y, y') = 0 \qquad (\beta = 1, \ldots, m < n + 1), \tag{1:7}$$

$$\psi_\mu\,[x_1, y(x_1), x_2, y(x_2)] = 0 \qquad (\mu = 1, \ldots, p \leqq 2n + 4) \tag{1:8}$$

one which minimizes a sum

$$J = g[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f[x, y(x), y'(x)]dx. \tag{1:9}$$

Here and elsewhere in the paper we use the notations y, $y(x)$ to represent the sets $(y_0, y_1, \ldots, y_n)$, $[y_0(x), y_1(x), \ldots, y_n(x)]$ respectively, with similar notations for sets of derivatives. The equations (1:7) and (1:8) correspond to the equations (1:3), (1:4), with $m = n$, $p = n + 4$ in the less general case. The function g is a more general form of the corresponding function for problem B, and the integral containing f does not appear in problem B.

2. The multiplier rule. Among the conditions which must be satisfied along a minimizing arc for the general problem of Bolza the first to be studied is usually the multiplier rule.

That rule may be stated as follows [VIII, p. 26]:

An admissible solution E of the equations $\phi_\beta = 0$, defined on an interval x_1x_2, is said to satisfy the multiplier rule if there exist constants λ_0, e_μ not all zero and a function

$$F = \lambda_0 f + \lambda_\beta(x)\,\phi_\beta \tag{2:1}$$

with multipliers $\lambda_\beta(x)$ continuous on x_1x_2 except possibly at values x defining corners of E where they have well-defined right and left limits, such that equations of the form

$$F_{y_i'} = \int_{x_1}^{x} F_{y_i}\,dx + c_i \tag{2:2}$$

are satisfied along E, and furthermore such that the equation

$$[(F - y_i'F_{y_i'})dx + F_{y_i'}dy_i]_{x_1}^{x_2} + \lambda_0 dg + e_\mu d\psi_\mu = 0 \tag{2:3}$$

holds at the ends of E for every choice of the differentials dx_1 dy_{i1}, dx_2, dy_{i2}. For an arc E satisfying the multiplier rule the multipliers λ_0, $\lambda_\beta(x)$ do not vanish simultaneously at any point of the interval x_1x_2. Every minimizing arc E for the problem of Bolza must satisfy the multiplier rule.

In the above paragraph and hereafter it is understood that the ranges of the subscripts are $i = 0, 1, \ldots, n$; $\beta = 1, \ldots, m$; and $\mu = 1, \ldots, p$, with $m = n$, $p = n + 4$ for the problem B. Terms with repeated indices will represent sums of such terms over a range of values associated with the given indices, as is customary in tensor analysis. The literal subscripts y_i, y_i' attached to F, or to other functions later on, indicate partial derivatives. Use is also made of the notation y_{i1} to designate $y_i(x_1)$, and other similar notations.

In applying the multiplier rule to problem B we may first notice that the function F for that problem takes the form

$$(2{:}4)\qquad F(x,\ y_0,\ y',\ \lambda) = \lambda_\beta(x)[h_\beta(x,\ y_0,\ y_0') - y_\beta'].$$

On imposing the condition that the function F shall satisfy the last n of the equations (2:2) along a minimizing arc E one finds that the multipliers have the values $\lambda_\beta(x) = c_\beta$. We shall accordingly hereafter designate the constant multipliers by λ_β. If we define a function $h(x,\ y_0,\ y_0',\ \lambda)$ by the equation

$$(2{:}5)\qquad h = \lambda_\beta h_\beta(x,\ y_0,\ y_0'),$$

then the first of the equations (2:2) may be written as

$$(2{:}6)\qquad h_{y_0'} = \int_{x_1}^{x} h_{y_0}\,dx + c_0.$$

The condition that the coefficients of the various differentials in (2:3) vanish is called the *transversality condition*. For the problem B, in which F is the function (2:4) and ψ_μ the conditions (1:4), it may be seen that the transversality condition on a minimizing arc E is equivalent essentially to the n equations

$$(2{:}7)\qquad -\lambda_\beta + \lambda_0 g_\beta = 0,$$

in which the subscript β attached to g indicates the partial derivative of g with respect to I_β.

Due to the peculiar nature of the problem B we may show that if a minimizing arc E for that problem satisfies the multiplier rule then E is *normal*, that is, there is no non-null set of multipliers λ_0, λ_β having $\lambda_0 = 0$ with which E satisfies the rule. This follows from the equation (2:7), since if $\lambda_0 = 0$ then also $\lambda_\beta = 0$, or all the multipliers λ_0, λ_β vanish. We summarize the last two paragraphs in the following theorem:

THEOREM 2:1. *Along a minimizing arc* E *for the problem* A *formulated in Section* 1 *there must exist a function* $h(x, y_0, y_0', \lambda)$

of the form (2:5), with constant multipliers l_β, such that the equation (2:6) is satisfied identically along E. The multipliers l_0, l_β must further satisfy with E the transversality conditions (2:7) in which the arguments of g are $y_{\beta 2}$, or the values of the integrals I_β taken along the arc E. The arc E for which these conditions are satisfied is, moreover, normal, so that the multiplier l_0 can always be taken to be unity.

The multiplier rule has three important corollaries which will be found useful in the succeeding discussion. Due to the special properties of our particular problem B the results stated in these corollaries appear here in a somewhat simpler form than they do in the theory of the more general problem C of Bolza [VIII, p. 27.1]. The first corollary contains the statement that at each point of an admissible arc E satisfying the equation (2:6) the function $h_{y_0'}$ has forward and backward derivatives, equal except possibly at corners, and such that the following Euler-Lagrange equation holds:

$$dh_{y_0'}/dx = h_{y_0} . \tag{2:8}$$

The second corollary provides us with the Weierstrass-Erdmann corner condition, which states that at each corner of an admissible arc E satisfying the equation (2:6) the function $h_{y_0'}$ has well-defined right and left limits which are equal. The third corollary, whose result is known as the Hilbert differentiability condition, implies that the functions $y_i'(x)$ defining an admissible arc E have continuous derivatives of at least the first order with respect to x on each sub-arc of E satisfying the equations (2:6) and $h_\beta - y_\beta' = 0$ on which the functions $y_i(x)$ defining E have continuous first derivatives and the function $h_{y_0' y_0'}$ is different from zero. The first two corollaries are immediate consequences of the equation (2:6). The proof of the third corollary involves an application of implicit function theory.

We have seen that a minimizing arc for problem B not only satisfies the equations $h_\beta - y_\beta' = 0$ but also must have associated with it constant multipliers λ_β with which it satisfies the equation (2:8). Solutions of these equations play an important role in the development of the theory. Bliss [VIII, pp. 33-39], and Bliss and Hestenes [V, pp. 309-311], have made studies of the solutions of the analogous sets of differential equations occurring in the more general theories of the problems of Bolza and Mayer, both of which apply to the problem B. However, there are some relations peculiar to problem B that make possible specialized results which we find it desirable to present. In order to study this phase of the problem we define an extremal to be an admissible arc $y_0(x)$, $y_\beta(x)$ and a set of constant multipliers λ_β, having continuous second derivatives and satisfying the equations

$$h_\beta - y_\beta' = 0, \qquad dh_{y_0'}/dx - h_{y_0} = 0.$$

A non-singular extremal is one along which the function $h_{y_0' y_0'}$ is different from zero. In order to discuss such extremals we introduce so-called canonical variables (x, y, z) related to the variables (x, y, y', λ) by the equations

$$\begin{aligned} z_0 &= F_{y_0'} = h_{y_0'}(x, y_0, y_0', \lambda), \\ z_\beta &= F_{y_\beta'} = -\lambda_\beta, \qquad 0 = h_\beta - y_\beta' \end{aligned} \tag{2:9}$$

whose determinant with respect to the variables (y', λ) has, except for a non-vanishing constant factor, the value $h_{y_0' y_0'}$. In a neighborhood of the elements (x, y, z) belonging to a non-singular extremal E the equations (2:9) have then a solution of the form

$$(2{:}10)\qquad \begin{aligned} y_0' &= P(x, y_0, z), \qquad y_\beta' = f_\beta(x, y_0, P),\\ \lambda_\beta &= -z_\beta = L_\beta(z_\beta), \end{aligned}$$

reducing to $y_i'(x)$, λ_β at each point $[x, y(x), z(x)]$ belonging to E. The function $P(x, y_0, z)$ is found by solving the first equation in (2:9) for y_0' and then substituting $-z_\beta$ for λ_β in the solution. The differential equations satisfied by the functions $y_i(x)$, $z_i(x)$ defining an extremal are

$$(2{:}11)\qquad \begin{aligned} y_0' &= P(x, y_0, z), & z_0' &= h_{y_0}(x, y_0, P, L),\\ y_\beta' &= h_\beta(x, y_0, P), & z_\beta' &= 0. \end{aligned}$$

These equations follow directly from the equations (2:10), the definition of the z's, and the Euler-Lagrange equation (2:8). The above equations have an interesting and useful homogeneity property. Since h is homogeneous of degree one in the λ's and the solutions (2:10) of equations (2:9) are unique, it follows that for an arbitrary multiplier k

$$P(x, y, kz) = P(x, y, z).$$

Hence if the functions $y_i(x)$, $z_i(x)$ are the solution of equations (2:11) uniquely determined by the initial values a_i, b_i at $x = x_1$, then the functions $y_i(x)$, $kz_i(x)$ will be the solutions corresponding to the initial values a_i, kb_i.

In the theory of the general problem C the extremals are discussed with the help of the canonical variables (x, y, z) defined by the equations

$$(2{:}12)\qquad z_i = F_{y_i'}, \qquad \phi_\beta = 0.$$

These equations have solutions in a neighborhood of a non-singular extremal of the form

(2:13) $$y_i' = P_i(x, y, z), \qquad \lambda_\beta = L_\beta(x, y, z).$$

The differential equations satisfied by the functions $y_i(x)$, $z_i(x)$ defining an extremal are then found to be the equations

(2:14) $$y_i' = P_i(x, y, z), \qquad z_i' = F_{y_i}[x, y, P(x,y,z), L(x,y,z)].$$

By means of these equations it is proved in the general theory [VIII, pp. 36-37] that every non-singular extremal arc E for a fixed value of the multiplier λ_0 is imbedded for values

(2:15) $$x_1 \leqq x \leqq x_2, \qquad a_{i0}, \qquad b_{i0} \qquad (i = 0, 1, \ldots, n)$$

in a (2n + 2)-parameter family of extremals defined by functions of the form

(2:16) $$y_i(x, a, b), \qquad z_i(x, a, b)$$

in which a, b are symbols for (n + 1)-dimensional sets $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$. The functions y_i, y_{ix}, z_i, z_{ix} have continuous partial derivatives of at least the second order in a neighborhood of the values (x, a, b) defining E, and the determinant

(2:17) $$\begin{vmatrix} y_{ia_k} & y_{ib_k} \\ z_{ia_k} & z_{ib_k} \end{vmatrix}$$

is different from zero along E. In terms of the variables (x, y, y', λ) the extremals (2:16) are defined by functions

(2:18) $$y_i(x, a, b), \qquad \lambda_\beta(x, a, b)$$

related to the functions (2:16) by the equations (2:12) and such that the functions y_i, y_{ix}, λ_β have continuous partial derivatives of at least the second order.

The analogous results for problem B are contained in the following theorem:

THEOREM 2:2. *For the problem* B, *which is equivalent to the problem* A, *the equations* (2:9), (2:10) *and* (2:11) *correspond respectively to the equations* (2:12), (2:13) *and* (2:14) *belonging to the general problem* C. *The functions* (2:16) *defining a* (2n + 2)-*parameter family of extremals in which a given non-singular extremal* E *is imbedded may be written for the problem* B *in the form*

$$(2{:}19)\qquad \begin{aligned} &y_0(x,\ a_0,\ b_0, \ell), && z_0(x,\ a_0,\ b_0, \ell),\\ &y_\beta(x,\ a_0,\ b_0, \ell) + a_\beta, && z_\beta = -\ell_\beta, \end{aligned}$$

in which the functions y_0, z_0 *are solutions of the first two equations in* (2:11) *with the variables* z_β *regarded as constant parameters, and in which*

$$y_\beta(x,\ a_0,\ b_0, \ell) = \int_{x_1}^{x} h_\beta(x,\ y_0,\ P)dx + a_\beta .$$

The functions y_i *have the homogeneity property*

$$y_i(x,\ a_0,\ kb_0,\ k\ell) = y_i(x,\ a_0,\ b_0, \ell).$$

The determinant (2:17) *has the form*

$$(2{:}20)\qquad \begin{vmatrix} y_{0a_0} & 0 & y_{0b_0} & y_{0\ell_\gamma} \\ y_{\beta a_0} & \delta_{\beta\gamma} & y_{\beta b_0} & y_{\beta\ell_\gamma} \\ z_{0a_0} & 0 & z_{0b_0} & z_{0\ell_\gamma} \\ 0 & 0 & 0 & -\delta_{\beta\gamma} \end{vmatrix}$$

$$(\gamma, \beta = 1,\ \ldots,\ n; \quad \delta_{\beta\gamma} = 0 \text{ \textit{if} } \gamma \neq \beta; \quad \delta_{\beta\beta} = 1).$$

This determinant is different from zero along E. *In terms of the variables* $(x,\ y,\ y', \ell)$ *the imbedding family* (2:19) *is defined by*

$$(2{:}21)\qquad y_0(x,\ a_0,\ b_0,\ \lambda),\qquad y_\beta(x,\ a_0,\ b_0,\ \lambda) + a_\beta.$$

On account of their homogeneity property the functions $y_i(x,\ a_0,\ b_0,\ \lambda)$ *depend only upon the parameter* a_0 *and the* n *ratios of the parameters* $b_0,\ \lambda_\beta$. *The functions appearing in* (2:19) *and* (2:21) *have continuity properties like those of the corresponding functions* (2:16) *and* (2:18).

3. *The necessary conditions of Weierstrass and Clebsch.* Two further conditions playing important roles in the theory of the problem of Bolza are the necessary conditions of Weierstrass and Clebsch. These conditions may be stated for problem C as follows [VIII, pp. 63 and 65]:

An admissible arc E satisfying the equations $\phi_\beta = 0$ and the multiplier rule, with multipliers $\lambda_0 = 1,\ \lambda_\beta(x)$, is said to satisfy the *necessary condition of Weierstrass* with these multipliers if the condition

$$(3{:}1)\qquad E(x,\ y,\ y',\ \lambda,\ Y') \geqq 0,$$

where

$$E = F(x,\ y,\ Y',\lambda) - F(x,\ y,\ y',\lambda) - (Y_i' - y_i')F_{y_i'}(x,\ y,\ y',\lambda),$$

is valid at every element $(x,\ y,\ y',\lambda)$ of E for all admissible sets $(x,\ y,\ Y') \neq (x,\ y,\ y')$ satisfying the equations $\phi_\beta = 0$. The arc E is further said to satisfy the *necessary condition of Clebsch* with the multipliers $\lambda_0 = 1,\ \lambda_\beta(x)$ if the condition

$$(3{:}2)\qquad F_{y_i' y_k'}(x,\ y,\ y',\lambda)\pi_i\pi_k \geqq 0 \qquad (i,\ k = 0,\ 1,\ \ldots,\ n)$$

is valid at every element $(x,\ y,\ y',\lambda)$ of E for all sets $(\pi_0,\ \ldots,\ \pi_n) \neq (0,\ \ldots,\ 0)$ satisfying the equations

$$(3{:}3)\qquad \phi_{\beta y_i'}(x,\ y,\ y',\lambda)\pi_i = 0.$$

Every normal minimizing arc for the problem of Bolza must satisfy these two conditions.

For the problem B the function E appearing in (3:1) takes the form

$$(3:4)\quad \begin{aligned} E(x, y_0, y_0', Y_0', \ell) &= h(x, y_0, Y_0', \ell) - h(x, y_0, y_0', \ell) \\ &\quad - (Y_0' - y_0')h_{y_0'}(x, y_0, y_0', \ell), \end{aligned}$$

as one readily verifies by use of the equations (2:4) and (2:5). The function E is here free from the arguments y_β, y_β', Y_β'. If it vanishes only for sets (x, y_0, Y_0') which are equal to (x, y_0, y_0'), it will also vanish only for full sets (x, y, Y') which are equal to (x, y, y'). This follows from the fact that the equations $\phi_\beta = 0$ for the problem B are the equations $h_\beta - y_\beta' = 0$, and hence the set (x, y, Y') is different from the set (x, y, y') if and only if the set (x, y_0, Y_0') is different from the set (x, y_0, y_0'). We see then that the Weierstrass condition for the problem B may be expressed in terms of the (xy_0)-projection of the minimizing arc E. A similar remark holds also for the Clebsch condition since for problem B the equations (3:3) are

$$h_{\beta y_0'}(x, y_0, y_0', \ell)\pi_0 - \pi_\beta = 0 \qquad (\beta = 1, \ldots, n).$$

Therefore the condition $(\pi_0, \ldots, \pi_n) \neq (0, \ldots, 0)$ implies $\pi_0 \neq 0$. By the use of equations (2:4) and (2:5) one finds that for problem B the inequality (3:2) is

$$h_{y_0' y_0'}(x, y_0, y_0', \ell)\pi_0^2 \geqq 0.$$

Since $\pi_0 \neq 0$, the Clebsch condition implies that along every minimizing arc for problem B that satisfies the multiplier rule the inequality $h_{y_0' y_0'} \geqq 0$ must be satisfied. The word normal appearing in the statement of the Weierstrass and Clebsch conditions may be deleted in the case of problem B, since for that problem arcs satisfying the multiplier rule are all normal. The results of this section are summarized for the problem A in the following theorem:

THEOREM 3:1. *Every minimizing arc* E *for the problem* A, *where* E *is an admissible arc satisfying the multiplier rule with multipliers* $\ell_0 = 1,\ \ell_\beta$, *must satisfy with these multipliers the inequality*

$$E(x, y_0, y_0'; Y_0', \ell) \geqq 0$$

at every element (x, y_0, y_0', ℓ) *of* E *for all admissible sets* $(x, y_0, Y_0') \neq (x, y_0, y_0')$. *The function* $E(x, y_0, y_0', Y_0', \ell)$ *is the function defined by the equation* (3:4). *Further, along the minimizing arc* E *the inequality*

$$h_{y_0'y_0'}(x, y_0, y_0', \ell) \geqq 0 \tag{3:5}$$

must be satisfied.

4. *The second variation and the accessory minimum problem.* Another condition which must necessarily be satisfied by a minimizing arc for the problem of Bolza is the condition of Mayer, two forms of which will be described in the next section. The condition of Mayer appears in the literature as a criterion for ensuring the non-negativeness of the so-called second variation. In order to discuss the second variation, we first define a set of *admissible variations* as a set of functions $\eta_i(x)$ which have derivatives and continuity properties like those of the functions $y_i(x)$. For the problem C the *second variation* $J_2(\xi, \eta)$ of J along E may then be defined to be

$$J_2 = 2\gamma\,[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] + \int^{x_2} 2\,\omega(x, \eta, \eta')\,dx, \tag{4:1}$$

where ξ_1, ξ_2 are constants. The term 2γ outside the integral sign is found by substituting

$$dx_s = \xi_s, \qquad dy_{is} = y_i'(x_s)\xi_s + \eta_i(x_s) \qquad (s = 1, 2)$$

in the quadratic form

$$2\gamma = \left[(F - y_i'F_{y_i})dx + 2F_{y_i}dy_i dx\right]_1^2 + 2q + 2e_\mu q_\mu$$

in which the constants e_μ are those of the multiplier rule and $2q$ and $2q_\mu$ are the quadratic forms in dx_s, dy_{is} whose coefficients are the second derivatives of g and ψ_μ, respectively. The integrand function $2\omega(x, \eta, \eta')$ is the quadratic form

$$2\omega = F_{y_i y_k}\eta_i\eta_k + 2F_{y_i y_k'}\eta_i\eta_k' + F_{y_i' y_k'}\eta_i'\eta_k'$$
$$(i, k = 0, 1, \ldots, n).$$

The equations

$$(4:2) \qquad \Phi_\beta(x, \eta, \eta') = \phi_{\beta y_i}\eta_i + \phi_{\beta y_i'}\eta_i' = 0,$$

$$(4:3) \qquad \Psi_\mu[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] = \psi_{\mu x_s}\xi_s + \psi_{\mu y_{is}}[y_i'(x_s)\xi_s + \eta_i(x_s)] = 0$$

are called the equations of variation. From the theory of the problem C we now quote the following theorem [VIII, p. 72]:

For every normal non-singular minimizing arc E without corners the second variation $J_2(\xi, \eta)$ in equation (4:1) must be positive or zero for all admissible sets of variations $\eta_i(x)$ and constants ξ_1, ξ_2 which satisfy the equations of variation $\Phi_\beta = \Psi_\mu = 0$ along E.

For the problem B the equations of variation and the second variation take on a somewhat simpler appearance. If the functions $h_\beta - y_\beta'$ are substituted for the functions ϕ_β in the equations (4:2), we find the equations

(4:4) $$h_{\beta y_0}\eta_0 + h_{\beta y_0'}\eta_0' - \eta_\beta{}' = 0 \qquad (\beta = 1, \dots, n).$$

By use of the quations (1:4) it may be seen that the quations (4:3) reduce for problem B to the n + 4 equations

(4:5) $$\xi_s = \eta_0(x_s) = \eta_\beta(x_1) = 0 \qquad (s = 1, 2;\ \beta = 1, \dots, n).$$

By making use of the specialized functions (1:5) and (2:5) and the linearity of the end conditions ψ_μ in (1:4), plus the fact that $\xi_1 = \xi_2 = 0$, one finds that the quadratic forms 2γ and 2ω are, for the problem B,

(4:6) $$2\gamma[\eta(x_2)] = g_{\beta\gamma}\,\eta_\beta(x_2)\,\eta_\gamma(x_2) \qquad (\gamma, \beta = 1, \dots, n),$$

(4:7) $$2\omega[x, \eta_0, \eta_0'] = h_{y_0y_0}\eta_0^2 + 2h_{y_0y_0'}\eta_0\eta_0' + h_{y_0'y_0'}\eta_0'^2.$$

These remarks we extend in the following theorem:

THEOREM 4:1. *For every non-singular minimizing arc* E *without corners the second variation for the problem* B *or* A *is*

(4:8) $$J_2(\eta) = g_{\beta\gamma}\,\eta_\beta(x_2)\,\eta_\gamma(x_2) + \int_{x_1}^{x_2}[h_{y_0y_0}\eta_0^2 + 2h_{y_0y_0'}\eta_0\eta_0' + h_{y_0'y_0'}\eta_0'^2]dx$$

where

(4:9) $$\eta_\beta(x) = \int_{x_1}^{x}(h_{\beta y_0}\eta_0 + h_{\beta y_0'}\eta_0')dx \qquad (\beta = 1, \dots, n).$$

This variation must be positive or zero for every admissible variation $\eta_0(x)$ *satisfying the equations* $\eta_0(x_1) = \eta_0(x_2) = 0$.

This result is an easy consequence of the application to problem B of the corresponding theorem described above for problem C.

In the theory of the problem C the fact that the second variation must be positive or zero leads to an auxiliary minimum problem called the *accessory minimum problem*. This is the problem of finding in the class of admissible sets of variations ξ_1, ξ_2, $\eta_i(x)$ satisfying the equations of variation $\Phi_\beta = 0$, $\Psi_\mu = 0$ along E, one which minimizes the second variation $J_2(\xi, \eta)$. The differential equations associated with this problem are

$$(4;10) \qquad d\Omega_{\eta_i'}/dx - \Omega_{\eta_i} = 0, \qquad \Phi_\beta = 0,$$

where Ω is the homogeneous quadratic form in the variables η_i, η_i', λ_β defined by the equation

$$(4;11) \qquad \Omega(x, \eta, \eta', \lambda) = \omega + \lambda_\beta \Phi_\beta .$$

These equations are analogous to the equations $\phi_\beta = 0$ and $dF_{y_i'}/dx = F_{y_i}$, the latter set of equations being derived from the equations (2;2) by differentiation of both members in each equation. The equations (4;10) are called the accessory differential equations. The transversality condition for the accessory minimum problem, analogous to the condition (2;3), is known as the accessory transversality condition.

By an accessory extremal is meant a solution $\eta_i(x)$, $\lambda_\beta(x)$ $(x_1 \leqq x \leqq x_2)$ of equations (4;10) having continuous derivatives η_i', η_i'', λ_β' on x_1x_2. A study of these extremals has been made by Bliss [VIII, pp. 73-76]. He introduces canonical variables (x, η_i, ζ_i) related to the variables $(x, \eta_i, \eta_i', \lambda_\beta)$ by means of the equations

$$(4;12) \qquad \zeta_i = \Omega_{\eta_i'}(x, \eta, \eta', \lambda), \qquad \Phi_\beta(x, \eta, \eta') = 0.$$

Bliss has shown that if E is a non-singular extremal then the equations (4;12) with $\lambda_0 = 1$ can be solved for the variables η_i', λ_β. These equations are of the form

$$(4;13) \qquad \eta_i' = \Pi_i(x, \eta, \zeta), \qquad \lambda_\beta = \Lambda_\beta(x, \eta, \zeta).$$

The equations (4:10) with $\lambda_0 = 1$ are then found to be equivalent to canonical accessory equations of the form

$$(4:14) \qquad \eta_i' = \mathcal{H}_{\zeta_i}(x, \eta, \zeta), \qquad \zeta_i' = -\mathcal{H}_{\eta_i}(x, \eta, \zeta),$$

in which $\mathcal{H}(x, \eta, \zeta)$ is the homogeneous quadratic function in the variables η, ζ defined by the formula

$$(4:15) \qquad \mathcal{H}(x, \eta, \zeta) = \zeta_i \pi_i - \Omega(x, \eta, \pi, \Lambda).$$

A fundamental set of 2n + 2 linearly independent solutions of equations (4:14) is given by the elements of the columns of the determinant (2:17), with $\eta_i = y_{ia_k}$, $\zeta_i = z_{ia_k}$ [VIII, p. 76].

For the problem B the function Ω has the form

$$(4:16) \qquad \begin{aligned} \Omega(x, \eta_0, \eta', \lambda) &= \omega(x, \eta_0, \eta_0') \\ &\quad + \lambda_\beta [h_{\beta y_0} \eta_0 + h_{\beta y_0'} \eta_0' - \eta_\beta'], \end{aligned}$$

where ω is the function defined in (4:7). The multipliers λ_β occurring here are constants. This follows from the fact that Ω does not contain η_β, so that the accessory equations (4:10) yield $d(\Omega_{\eta_\beta'})/dx = d(-\lambda_\beta)/dx = 0$ for the problem B. If in the equations (2:3) we replace the function F by the function Ω defined in (4:16), the function g by the quadratic form 2γ defined in (4:6), and the functions ψ_μ by the functions Ψ_μ defined in (4:5), we find that the accessory transversality condition for the problem B is equivalent essentially to the n equations

$$(4:17) \qquad \zeta_\beta + g_{\beta\gamma} \eta_\gamma(x_2) = 0 \qquad (\beta, \gamma = 1, \ldots, n),$$

where the $\zeta_\beta = \Omega_{\eta_\beta'} = -\lambda_\beta$ are constants along a given accessory extremal. According to Theorem 2:2 the determinant (2:20) is for problem B the determinant (2:17) of problem C. The elements of the columns of (2:20) therefore form a fundamental set of solutions

η_1, ζ_1 of the canonical accessory equations (4:14) calculated for problem B. These equations will be useful in the next section. To develop them, we notice that for problem B the equations (4:12) are

$$(4{:}18)\qquad \begin{gathered} \zeta_0 = h_{y_0 y_0'}\eta_0 + h_{y_0' y_0'}\eta_0' + \lambda_\beta h_{\beta y_0'}, \qquad \zeta_\beta = -\lambda_\beta \\ h_{\beta y_0}\eta_0 + h_{\beta y_0'}\eta_0' - \eta_\beta' = 0, \end{gathered}$$

so that in the solutions (4:13) the functions Π_1, Λ_β have the values

$$(4{:}19)\qquad \begin{gathered} \Pi_0(x, \eta_0, \zeta) = [\zeta_0 - h_{y_0 y_0'}\eta_0 + \zeta_\beta h_{\beta y_0'}]/h_{y_0' y_0'}, \\ \Pi_\beta(x, \eta_0, \zeta) = h_{\beta y_0}\eta_0 + h_{\beta y_0'}\Pi_0, \qquad \Lambda_\beta = -\zeta_\beta. \end{gathered}$$

The function $\mathcal{H}$ defined in (4:15) is then for problem B

$$(4{:}20)\qquad \mathcal{H}(x, \eta_0, \zeta) = \zeta_0\Pi_0 + \zeta_\beta[h_{\beta y_0}\eta_0 + h_{\beta y_0'}\Pi_0] - \omega(x, \eta_0, \Pi_0),$$

in which the function ω has the definition (4:7). With this value for the function $\mathcal{H}$ the equations (4:14) become, in detail,

$$(4{:}21)\qquad \begin{gathered} \eta_0' = \Pi_0(x, \eta_0, \zeta), \qquad \zeta_0' = -\zeta_\beta h_{\beta y_0} + h_{y_0 y_0}\eta_0 + h_{y_0 y_0'}\Pi_0, \\ \eta_\beta' = h_{\beta y_0}\eta_0 + h_{\beta y_0'}\Pi_0, \qquad \zeta_\beta' = 0. \end{gathered}$$

One readily verifies that one solution of the equations (4:21) is

$$(4{:}22)\qquad \eta_1 \equiv 0, \qquad \zeta_0 = h_{y_0'}, \qquad \zeta_\beta = -l_\beta,$$

where the constants l_β are the multipliers belonging to E. We summarize the foregoing statements in the following theorem:

THEOREM 4:2. *The canonical accessory equations for the problem B, which is equivalent to problem A, are the equations (4:21). One may solve the first two of these equations for* η_0, ζ_0 *as functions of x and the parameters* ζ_β, *from which it follows that the solutions* η_β *are of the form*

$$(4{:}23) \qquad \eta_\beta(x) = \int_{x_1}^{x} [h_{\beta y_0}\eta_0 + h_{\beta y_0'}\eta_0']dx + k_\beta,$$

where the k_β *are constants. The columns of the determinant* (2:20) *form a complete set of solutions of the equations* (4:21); *one solution in particular is given in* (4:22). *The accessory transversality conditions for the problem* B, *or* A, *are given by the equations* (4:17), *in which the functions* η_β *and the constants* $\zeta_\beta = -\lambda_\beta$ *are solutions of the equations* (4:21).

5. *The condition of Mayer*. In the literature there appear four closely related forms of the so-called condition of Mayer for the problem of Bolza in the calculus of variations. These are all conditions necessary for the non-negativeness, and in a strengthened form sufficient for the positiveness, of the second variation, as stated for problem B in Theorem 4:1. Of these four forms we shall make use of but one, that one being due to Bliss. For the less general problem of Bolza in which one end point of the admissible arcs is fixed Hestenes has defined a focal point condition which is also a condition of Mayer. Since the criterion for a focal point is comparatively simple theoretically, and since the problem B is a problem of Bolza of the type for which a focal point condition is defined, we also discuss that condition in this section.

First, we shall have need of a few definition. By the *order* q *of abnormality of* E *on an interval* x'x" is meant the number q of linearly independent sets of multipliers of the form $l_0 = 0$, $l_\beta(x)$ with which E satisfies the conditions (1:7) and (2:2) on x'x". We next define a function Q(z) in terms of the second variation. After the usual integration by parts the second variation $J_2(\xi, \eta)$ along an accessory extremal is expressible as a quadratic form in ξ_1, ξ_2 and the end values of the functions $\eta_i(x)$, $\zeta_i(x)$ defining the extremal. Let $\xi_{1\tau}$, $\xi_{2\tau}$ $\eta_{i\tau}$, $\zeta_{i\tau}$ ($\tau = 1, \ldots, t$) be a maximum set of linearly independent constants and accessory extremals containing only sets ξ_1, ξ_2, η_i, ζ_i which are linearly independent of the special ones for which $\xi_1 = 0$, $\xi_2 = 0$, $\eta_i(x) \equiv 0$, and which satisfy the equations $\Psi_\mu = 0$. For the set

$$(5{:}1) \qquad \xi_1 = \xi_{1\tau}z_\tau, \qquad \xi_2 = \xi_{2\tau}z_\tau, \qquad \eta_i = \eta_{i\tau}z_\tau, \quad \zeta_i = \zeta_{i\tau}z_\tau$$

the second variation may then be expressed as the quadratic form

$$(5{:}2) \qquad Q(z) = J_2(\xi_\tau z_\tau, \eta_\tau z_\tau)$$

in the constants $(z_1, \ldots, z_t)$. A value $x_3 \neq x_1$ is said to define a point 3 conjugate to 1 if there exists an accessory extremal η_i, ζ_i satisfying the equations $\eta_i(x_1) = 0$, having $\eta_i(x_3) = 0$, and having the set $(\eta) \not\equiv 0$ on x_1x_3. It is known [VII, p. 804, Theorem 5:1] that points 3 conjugate to 1 are the points other than 1 at which the matrix

$$(5{:}3) \qquad \left\| \begin{matrix} \eta_{ij}(x_3) \\ \eta_{ij}(x_1) \end{matrix} \right\|$$

has rank less than $2n + 2 - q$, where η_{ij} forms with ζ_{ij} ($i = 0, 1, \ldots, n$; $j = 1, \ldots, 2n + 2$) a complete set of solutions of the canonical accessory equations, and q is the number of linearly independent sets η_i, ζ_i having $\eta_i(x) \equiv 0$ on x_1x_3. This number q is also the order of abnormality of E on the interval x_1x_3. We now quote from the theory for problem C [VII, p. 802, Theorem 4:3] the result that if the extremal E is non-singular and the second variation J_2 is non-negative along E, then the condition $Q(z) \geqq 0$ must hold for all sets of constants z not all zero;

moreover, if the order of abnormality of E is the same on every sub-interval x_3x_2 of x_1x_2, then there can be no point 3 conjugate to 1 on E between its end points 1 and 2. This condition may be called the condition of Mayer as formulated by Bliss for the general problem C of Bolza.

If for problem C stated in Section 1 the first (n + 1) of the equations $\psi_\mu = 0$ fix the coordinates of the initial points of admissible arcs at x_1 and the remaining (p - n - 1) of the equations are conditions only on the coordinates of the end points of the arcs at x_2, then the problem is called a problem of Bolza with second end point variable. For this problem a value $x_3 \neq x_2$ is said to define a focal point 3 of the manifold $\psi_\nu = 0$ ($\nu = n + 2, \ldots, p$) on the extremal arc E if there exists an accessory extremal η_i, ζ_i satisfying with a constant ξ_2 the equations

$$(5{:}4) \qquad \Psi_\nu = 0 \qquad (\nu = n + 2, \ldots, p)$$

and the accessory transversality condition at x_2, having $\eta_i(x_3) = 0$, and having the set $(\eta) \not\equiv (0)$ on x_3x_2. From the theory of problem C [X, p. 545] we know that on a non-singular extremal arc E for which the second variation J_2 is non-negative and for which the abnormality of E is the same on every sub-interval of x_1x_2, there can be no focal point 3 of the manifold $\psi_\nu = 0$ ($\nu = n + 1, \ldots, p$) between the end points 1 and 2 of E. This may be called the focal point condition for a problem of Bolza with second end point variable.

In order to apply these results to problem B we first note that this problem is one with only the second end point variable. The equations (5:4) for problem B are simply

$$(5{:}5) \qquad \xi_2 = 0, \qquad \eta_0(x_2) = 0,$$

as we see from equations (4:5). To calculate $Q(z)$ as defined in (5:2) we note that for an accessory extremal η_i, ζ_i satisfying the conditions (4:5), and with the function ω defined in (4:7), one finds

$$\int_{x_1}^{x_2} 2\omega\, dx = \int_{x_1}^{x_2} 2\Omega\, dx = \eta_i\zeta_i\Big|_1^2 = \eta_\beta(x_2)\zeta_\beta .$$

Hence the value of $J_2(\eta)$, defined in (4:8), for the set (5:1) is

$$Q(z) = z_\sigma z_\tau[g_{\beta\gamma}\eta_{\beta\sigma}(x_2)\,\eta_{\gamma\tau}(x_2) + \zeta_{\beta\sigma}\eta_{\beta\tau}(x_2)] \tag{5:6}$$
$$(\sigma, \tau = 1, \ldots, t \leqq n - 1).$$

The inequality $t \leqq n - 1$ follows from the fact, noted in Theorem 4:2, that at least one of the (2n + 2) independent solutions of the canonical accessory equations has $\eta_i(x) = 0$. To determine the set (5:1) the n + 2 conditions on the end values of the functions $\eta_i(x)$, given in (4:5), must be impressed upon the linear family, containing 2n + 1 or fewer parameters, including all the extremals which have the property that their functions $\eta_i(x)$ are not all identically zero. We now apply Theorem 4:2 and the results quoted in the preceding paragraphs to get the following theorem concerning the problems B and A.

THEOREM 5:1. *Let* E_{12} *be a non-singular extremal arc for the problem* A *along which the second variation* $J_2(\eta)$ *in* (4:8) *is non-negative, and let the order of abnormality* q *of* E *be the same on every sub-interval of* x_1x_2. *Then the two following conditions must hold:*

(1) *There can be no point* 3 *conjugate to the initial point* 1 *between* 1 *and* 2 *on the arc* E_{12}, *and the condition* $Q(z) \geqq 0$ *must be satisfied for all values of the constants* z *not all zero, where*

$$Q(z) = z_\sigma z_\tau [g_{\beta\gamma} \eta_{\beta\sigma}(x_2) \eta_{\gamma\tau}(x_2) - \lambda_{\beta\sigma} \eta_{\beta\tau}(x_2)] \tag{5:7}$$
$$(\beta, \gamma = 1, \ldots, n;\ \sigma, \tau = 1, \ldots, t \leqq n - 1).$$

In this expression

$$\eta_{\beta\sigma}(x) = \int_{x_1}^{x} [h_{\beta y_0} \eta_{0\sigma} + h_{\beta y_0'} \eta_{0\sigma}'] dx \tag{5:8}$$

and the functions $\eta_{0\sigma}(x)$ $(\sigma = 1, \ldots, t)$ *form with the constants* $\lambda_{\beta\sigma}$ *a maximal set of solutions of the equations*

$$\begin{gathered} d\omega_{\eta_0'}/dx - \omega_{\eta_0} + \lambda_\beta (dh_{\beta y_0'}/dx - h_{\beta y_0}) = 0, \\ \eta_0(x_1) = \eta_0(x_2) = 0, \end{gathered} \tag{5:9}$$

for which the functions $\eta_{0\sigma}(x)$ *are linearly independent. A conjugate point* 3 *to* 1 *on* E_{12} *is a point, other than* 1, *at which the matrix*

$$\left\| \begin{matrix} \eta_{0\pi}(x_1) \\ \eta_{0\pi}(x_3) \\ \eta_{\beta\pi}(x_3) \end{matrix} \right\| \tag{5:10}$$
$$(\beta = 1, \ldots, n;\ \pi = 1, \ldots, t + 2)$$

is of rank less than t + 2, *the set* $\eta_{0\pi}$ $(\pi = 1, \ldots, t + 2)$ *being a maximal set of solutions of the equation* (5:9) *for which the functions* $\eta_{0\pi}(x)$ *are linearly independent, and the functions* $\eta_{\beta\pi}$ *being defined by equations analogous to* (5:8).

(2) *There can be no focal point* 3 *of the manifold* $x_2 - a_2 = y_0(x_2) = 0$ *on* E_{12} *between its end points* 1 *and* 2. *The focal points are defined by values* x_3 *for which the determinant*

$$\left| \begin{matrix} \eta_{0\rho}(x_3) \\ \eta_{0\rho}(x_2) \\ g_{\beta\gamma}[\eta_{\gamma\rho}(x_3) - \eta_{\gamma\rho}(x_2)] + \lambda_{\beta\rho} \end{matrix} \right| \tag{5:11}$$
$$(\beta, \gamma = 1, \ldots, n;\ \rho = 1, \ldots, n + 2)$$

vanishes. Here the sets $\eta_{0\rho}(x)$, $\lambda_{\beta\rho}$ $(\rho = 1, \ldots, n + 2)$ *are a maximal set of linearly independent solutions of the equation* (5:9), *and the functions* $\eta_{\gamma\rho}(x)$ *are defined by equations analogous to* (5:8).

According to Theorem 4:2 the columns of the determinant (2:20) form a complete set η_{ij}, ζ_{ij} $(i = 0, 1, \ldots, ; j = 1, \ldots, 2n + 2)$ of solutions of the canonical accessory equations (4:21). By deleting from that set the members linearly dependent upon sets for which $\eta_i(x) \equiv 0$, and by making use of the special nature of the set, one may readily relate the rank of the matrix (5:3) to that of the matrix (5:10) in such a way as to give the result stated in the theorem. To finish our discussion of the theorem we need to establish further only the fact that a focal point as defined in this section for problem B, or A, is a point at which the determinant (5:11) vanishes. To this end we desire to show that a value $x = x_3$ for which the determinant

$$(5:12) \qquad \begin{vmatrix} \eta_{ij}(x_3) \\ \eta_{0j}(x_2) \\ g_{\beta\gamma}\eta_{\gamma j}(x_2) + \zeta_{\beta j} \end{vmatrix}$$

$$(i = 0, \ldots, n; \; j = 1, \ldots, 2n + 2; \; \beta, \gamma = 1, \ldots, n)$$

vanishes determines a focal point 3 of the manifold $x_2 - a_2 = y_0(x_2) = 0$. In this determinant the functions η_{ij}, ζ_{ij} are understood to form a complete set of solutions of the canonical accessory equations (4:21). First, let us suppose that the determinant (5:12) vanishes at a value $x_3 \neq x_2$. Then there exist constants a_j not all zero such that

$$
\begin{aligned}
&a_j\eta_{ij}(x_3) = 0,\\
(5{:}13)\qquad &a_j\eta_{0j}(x_2) = 0,\\
&a_j[g_{\beta\gamma}\eta_{\gamma j}(x_2) + \zeta_{\beta j}] = 0.
\end{aligned}
$$

It follows that $\eta_i = a_j\eta_{ij}$, $\zeta_i = a_j\zeta_{ij}$ is a set of functions satisfying the conditions

$$(5{:}14)\qquad \eta_0(x_2) = 0, \quad g_{\beta\gamma}\eta_\gamma(x_2) + \zeta_\beta = 0 \quad (\beta, \gamma = 1, \ldots, n),$$

which, along with $\xi_2 = 0$, are the equations (5:4) and the accessory transversality condition. Further, η_i, ζ_i so defined are such that the functions η_i vanish at x_3, but do not vanish identically on x_3x_2. For if $\eta_i \equiv 0$ on x_3x_2 we have $\zeta_\beta = 0$ also, since the set η_β, ζ_β satisfies the second of equations (5:14). One may verify by examining the canonical accessory equations (4:21) that then $\zeta_0(x) \equiv 0$ on x_3x_2. Hence we would have $\eta_i = a_j\eta_{ij} \equiv 0$, $\zeta_i = a_1\zeta_{ij} \equiv 0$ on x_1x_2, which contradicts the assumption that the functions η_{ij}, ζ_{ij} form a complete set. By definition the point 3 is then a focal point to the given manifold. Conversely, a value x_3 for which a non-identically zero set η_i vanishes, where η_i, ζ_i satisfy the equations (5:14), is a value for which the determinant (5:12) vanishes. This follows from the fact that such a set is expressible as a linear combination $\eta_i = a_j\eta_{ij}$, $\zeta_i = a_j\zeta_{ij}$ of the complete set η_{ij}, ζ_{ij}. For the constants a_j, not all zero, thus determined the equations (5:13) hold and therefore the determinant (5:12) vanishes. That the determinant (5:11) vanishes when and only when the determinant (5:12) vanishes may be seen by making use of the special properties of the complete set η_{ij}, ζ_{ij} yielded by the determinant (2:20). For, if we let $\eta_{i\rho}$, $\zeta_{i\rho}$. ($\rho = 1, \ldots, n+2$) be a set linearly dependent on the set composed of the elements of the first column and the last n + 1 columns of the determinant (2:20), we may write the determinant (5:12) in the form

$$\begin{vmatrix} 0 & \eta_{0\rho}(x_3) \\ & \eta_{\beta\rho}(x_3) \\ 0 & \eta_{0\rho}(x_2) \\ g_{\beta\alpha}\delta_{\alpha\gamma} & g_{\beta\gamma}\eta_{\gamma\rho}(x_2) + \zeta_{\beta\rho} \end{vmatrix}$$

$$(\alpha, \beta, \gamma = 1, \ldots, n; \ \rho = 1, \ldots, n+2;$$
$$\delta_{\beta\beta} = 1; \ \ \delta_{\beta\gamma} = 0 \text{ if } \beta \neq \gamma\ ).$$

By substituting $\zeta_{\beta\rho} = -\lambda_{\beta\rho}$ and by suitably manipulating the rows of this determinant one derives the determinant (5:11).

As is readily seen from the preceding discussion, we have the following corollary to Theorem 5:1:

COROLLARY 5:1. *The* n + 2 *sets of functions* $\eta_{i\rho}$ *and the associated constants* $\lambda_{\beta\rho}$ *appearing in the condition* (2) *of Theorem* 5:1 *may be taken to be the functions and constants*

$$\eta_{i\gamma} = y_{il_\gamma}, \qquad \eta_{i,n+1} = y_{ia_0}, \qquad \eta_{i,n+2} = y_{ib_0},$$
$$\lambda_{\beta\gamma} = -\delta_{\beta\gamma}, \qquad \lambda_{\beta,n+1} = 0, \qquad \lambda_{\beta,n+2} = 0$$
$$(i = 0, 1, \ldots, n; \ \beta, \gamma = 1, \ldots, n),$$

in which the right members are the derivatives occurring in the columns of the determinant (2:20). *Suitable linear combinations of these derivatives may be used for the sets* $\eta_{i\pi}$ *and* $\eta_{i\sigma}$, $\lambda_{\beta\sigma}$ *occurring in condition* (1).

6. *Sufficiency theorems from the theory of the problem of Bolza.* Sufficiency theorems for the problem A will be stated in terms of strengthened forms of the necessary conditions discussed in the preceding sections. We use the Roman numeral I to designate

the multiplier rule discussed in Section 2. Similarly, the symbol II_N' is used to designate the *strengthened condition of Weierstrass*, which is defined as follows: An arc E is said to satisfy the condition II_N' if the inequality

$$E(x, y_0, y_0', Y_0', \ell) > 0$$

holds at every element (x, y_0, y_0', ℓ) in a neighborhood N of those belonging to E and for all admissible sets $(x, y_0, Y_0') \neq (x, y_0, y_0')$. For problem A the *strengthened condition of Clebsch*, denoted by III', is the condition that the inequality $h_{y_0' y_0'}(x, y_0, y_0', \ell) > 0$ shall hold along E. We define the *condition* IV' for problem A to be the condition (1) of Theorem 5:1 strengthened by the assumption that the quadratic form Q(z) is not only positive but is positive definite. Likewise, the *condition* IV_f' is defined to be the condition (2) of Theorem 5:1 strengthened by excluding the possibility of a focal point 3 being at the point 1. We have here defined the strengthened conditions for problem A, or B. Where similar strengthened conditions are defined for problem C, we have the following results from the theory of the problem of Bolza [VII, p. 816, Theorem 9:4; X, p. 546, Theorems 1 and 3]:

Let E be an admissible arc without corners satisfying the equations $\phi_\beta = \psi_\mu = 0$ for a problem of Bolza, and satisfying further the conditions I, II_N', III', and either IV' or IV_f' with a set of multipliers $\ell_0 = 1$, $\ell_\beta(x)$, e_μ. Then there exists a neighborhood V of E in xy-space and a neighborhood T of the space of points $(x_1, y_{i1}, x_2, y_{i2})$ such that $J(C) > J(E)$ for every admissible arc C in V with ends in T satisfying the equations $\phi_\beta = \psi_\mu = 0$ and not identical with E.

When this result is applied to problem B and interpreted for problem A one obtains the following theorem:

THEOREM 6:1. SUFFICIENT CONDITIONS FOR A RESTRICTED STRONG RELATIVE MINIMUM. *Let* E *be an admissible arc for the problem* A *such that* E *is without corners, joins two given fixed points in the* xy-*plane, and satisfies the conditions* I, II_N', III', *and either* IV' *or* IV_f' *with a set of multipliers* $\ell_0 = 1, \ell_\beta$. *Then there exists a neighborhood* V *of* E *such that* $g(C) > g(E)$ *for every admissible arc* C *in* V *not identical with* E, *joining the end points of* E, *and satisfying on* x_1x_2 *the condition that*

$$(6:1) \qquad \left|\int_{x_1}^{x} [h_\beta(C) - h_\beta(E)]dx\right| < \epsilon ,$$

where ϵ *is a sufficiently small positive constant*.

In the theorem it is understood that the functions $h_\beta(C)$, $h_\beta(E)$ are the functions $h_\beta(x, y, y')$ occurring in the equations (1:1), with arguments the functions y, y' defining C and E respectively. Similar notations are used in writing $I_\beta(E)$, $I_\beta(C)$, g(E) and g(C).

It is desirable to weaken the restrictive condition (6:1). In the next section we show that with an additional normality assumption we may weaken the condition (6:1) to the extent that it be required to hold only at $x = x_2$. Reflection on the possible nature of the function g shows that one could not in general remove the condition entirely.

The following sufficiency theorem for the accessory minimum problem, which was discussed in Section 4, will be found useful in the next section:

THEOREM 6:2. *Let* E *have the properties specified in Theorem* 6:1 *except possibly the property that it satisfies* II_N'. *Then along* E *the inequality* $J_2(\eta) > 0$ *holds for every admissible* variation $\eta_0(x)$ satisfying the equations $\eta_0(x_1) = \eta_0(x_2) = 0$ and not identically zero on x_1x_2.

For the proof of the theorem we shall refer to the literature of the problem C of Bolza. Results for problem A are then immediate by the use of Theorem 4:1. For the case when E satisfies IV' we refer to two theorems by Hestenes [VII, p. 811, Theorem 8:3, and X, p. 546, Theorem 3]. When E satisfies IV_f' we refer to three lemmas, again by Hestenes [X, pp. 549-550, Lemmas 8, 5, 6].

We note that the accessory minimum problem for problem **A** is a problem of the type **A**, since the functional to be minimized is a function of integrals.

7. Extension of the sufficiency proof. We have seen that the theory for the problem of Bolza yields a restricted relative minimum for problem A, and we noted that a weakening of the restrictive condition (6:1) on the admissible arcs is desirable. We may effect the desired weakening of that condition if E satisfies an additional normality condition, namely, that the arc E satisfies the Euler-Lagrange equation $dh_{y_0'}/dx - h_{y_0} = 0$ with only one set of multipliers when one keeps a non-zero multiplier fixed. Subarcs of E need not satisfy the normality condition. The theorem to be proved in this section may now be formulated as follows:

THEOREM 7:1. SUFFICIENT CONDITIONS FOR A STRONG RELATIVE MINIMUM. If the admissible arc E satisfies the hypotheses of Theorem 6:1 and in addition the normality condition defined above, then the conclusions of Theorem 6:1 hold with the restrictive condition (6:1) required to hold only at $x = x_2$.

The proof of the theorem will be based on a series of five lemmas, the last three of which are similar to lemmas used by Hestenes to prove an analogous result for the isoperimetric problem [XI]. In these lemmas it is understood that the arc E satisfies the hypotheses made in Theorem 7:1. As a preliminary to the first lemma we note that in Section 1 it was assumed that not all the first partial derivatives of g vanish on E. Without loss of generality we may assume that $g_1 \neq 0$. In fact we may suppose that $g_1 > 0$ since we may replace I_1 by $-I_1$ if necessary. We shall fix the multiplier $\ell_1 = g_1(E)$. With these statements in mind we formulate the following lemma:

LEMMA 7:1. *With the fixed relation* $\ell_1 = g_1(E)$ *the normality condition is equivalent to the condition that the equation*

$$[dh_{py_0'}/dx - h_{py_0}]a_p = 0 \qquad (p = 2, \dots, n) \tag{7:1}$$

holds along E *only in case the constants* a_p *are all zero.*

For, if two distinct sets of multipliers $\ell_1 = g_1(E)$, ℓ_p $(p = 2, \dots, n)$ and $\lambda_1 = g_1(E)$, λ_p satisfy the Euler-Lagrange equation with E so also does the non-null set $\ell_1 - \lambda_1 = 0$, $\ell_p - \lambda_p$. Then equation (7:1) holds with $a_p = \ell_p - \lambda_p$. Conversely, if a_p is a non-null set for which (7:1) holds on E, then the Euler-Lagrange equation is satisfied on E by the two distinct sets $\ell_1 = g_1(E)$, ℓ_p and $\ell_1 = g_1(E)$, $\ell_p + a_p$, where ℓ_1, ℓ_p are the multipliers belonging to E.

LEMMA 7:2. *If* $\eta_0(x') = \eta_0(x'') = 0$ *then*

$$\ell_\beta\,[\eta_\beta(x'') - \eta_\beta(x')] = 0 \qquad (x_1 \leqq x' \leqq x'' \leqq x_2) \tag{7:2}$$

holds for every admissible variation $\eta_i(x)$. *From this it follows that if* $\ell_1 \neq 0$, $\eta_1(x_1) = \eta_p(x_1) = \eta_p(x_2) = 0$ $(p = 2, \dots, n)$, *then* $\eta_1(x_2) = 0$ *also.*

That the equation (7:2) holds under the conditions stated in the lemma is readily provable by multiplying the equations of variation

$$h_{\beta y_0}\eta_0 + h_{\beta y_0'}\eta_0' - \eta_\beta' = 0 \qquad (\beta = 1, \dots, n)$$

by the multipliers ℓ_β, adding, and applying the usual integration by parts with the help of the equation $dh_{y_0'}/dx - h_{y_0} = 0$.

In the next lemmas there appears an integral J_ℓ defined by the equation

$$J_\ell = \int_{x_1}^{x_2} h(x, y_0, y_0', \ell)\,dx, \tag{7:3}$$

in which the integrand is the sum $\ell_\beta h_\beta$. A value $x_3 \neq x_1$ is said to define a point 3 conjugate to 1 on E relative to J_ℓ if there exists a solution $\eta(x)$ of the equation

$$d\,\omega_{\eta'}/dx - \omega_\eta = 0,$$

having $\eta(x_1) = \eta(x_3) = 0$ and $\eta \not\equiv 0$ on x_1x_3, where the multipliers ℓ_β appearing in J_ℓ and ω are those belonging to E. The function ω that occurs here is defined by equation (4:7).

LEMMA 7:3. HAHN'S LEMMA. *If E has on it no point 3 conjugate to 1 relative to J_ℓ, there exist neighborhoods V_1 of E in xy-space, N of the end values of E in $(x_1y_1x_2y_2)$-space and L of the multipliers ℓ_β belonging to E such that for every set $(x_1y_1x_2y_2)$ in N and ℓ_β in L there is an extremal E_ℓ in V_1 having $(x_1y_1x_2y_2)$ as its end values and ℓ_β as its multipliers. Moreover, for these values of ℓ_β the inequality $J_\ell(C) > J_\ell(E_\ell)$ holds for every admissible arc C in V_1 joining the ends of E_ℓ and not identical with E_ℓ.*

This lemma is analogous to one given by Hahn and has been established by Birkhoff and Hestenes [IX, pp. 253-254] following a method given by Bliss [IV, pp. 267-270] in the proof of a similar theorem for the problem of Bolza. The assumptions made in Theorem 7:1 do not exclude the possiblity of there being pairs of conjugate points on E relative to J_{ℓ}. However, the arc E can be subdivided into a finite number of subarcs on each of which there are no pairs of conjugate points relative to J_{ℓ} [cf. III, p. 739]. In this case we have the following extension of Lemma 7:3:

LEMMA 7:4. EXTENDED LEMMA OF HAHN. *Let* $t_0 = x_1 < t_1 < \cdots < t_{w+1} = x_2$ *be a set of values such that there is no pair of conjugate points on* E *relative to* J_{ℓ} *on any of the intervals* $t_{r-1} \leqq x \leqq t_r$ $(r = 1, \ldots, w + 1)$. *Then there exist neighborhoods* V_1 *of* E *in xy-space,* M *of the values* $(x, y) = (t_s, b_s)$ $(s = 1, \ldots, w)$ *belonging to* E *and* L *of the multipliers* ℓ_β *belonging to* E *such that for every set* (t_s, b_s) *in* M *and* ℓ_β *in* L *there is a broken extremal* E_{ℓ} *in* V_1 *joining the points* 1 *and* 2, *passing through the points* (t_s, b_s), *having no corners on the intervals* $t_{r-1} < x < t_r$, *and having* ℓ_β *as its multipliers. Moreover, for these values of* ℓ_β *the inequality* $J_{\ell}(C) > J_{\ell}(E_{\ell})$ *holds for every admissible arc* C *in* V_1 *joining the end points of* E_{ℓ}, *passing through the points* (t_s, b_s) *and not identical with* E_{ℓ}.

Lemma 7:4 follows as a result of applying Lemma 7:3 to the successive subarcs of E determined by the intervals $t_{r-1} \leqq x \leqq t_r$ $(r = 1, \ldots, w + 1)$.

LEMMA 7:5. *Let* $t_0, t_1, \ldots, t_{w+1}$ *be the set of values defined in Lemma 7:4. There exists a* $(w + n - 1)$*-parameter family of broken extremals*

$$\begin{gathered} y(x, b_1, \ldots, b_w, c_2, \ldots, c_n), \\ \ell_1 = g_1(E), \qquad \ell_p(b_1, \ldots, b_w, c_2, \ldots, c_n) \qquad (7{:}4) \\ (p = 2, \ldots, n;\ x_1 \leqq x \leqq x_2) \end{gathered}$$

containing E *for* $b_s = b_{s0}$, $c_p = c_{p0}$, *passing through the points* 1, 2 *and* $(x, y) = (t_s, b_s)$, *having no corners on the intervals* $t_{r-1} < x < t_r$, *and satisfying the conditions*

$$(7;5) \qquad \int_{x_1}^{x_2} h_p(x, y, y')dx = c_p.$$

The functions $y(x, b, c)$, $y_x(x, b, c)$, $\ell_p(b, c)$ *have continuous first and second derivatives for values* (x, b, c) *near those belonging to* E *except possibly at the corner points.*

In order to establish this lemma, let

$$(7;6) \quad Y(x, b_1, \ldots, b_w, \ell_2, \ldots, \ell_n),\ \ell_1 = g_1(E),\ \ell_p \quad (x_1 \leqq x \leqq x_2)$$

be a $(w + n - 1)$-parameter family of broken extremals containing E for values $b_s = b_{s0}$, $\ell_p = \ell_{p0}$, passing through the points 1, 2 and $(x, y) = (t_s, b_s)$, and having no corners on the intervals $t_{r-1} < x < t_r$. Except at corner points the functions $Y(x, b, \ell)$ $Y_x(x, b, \ell)$ have continuous first and second derivatives for values (x, b, ℓ) near those belonging to E. The existence of a family of this type follows readily from the first part of Lemma 7;4 and our earlier studies, in Section 2, of the properties of functions defining extremals. When the function $Y(x, b, \ell)$ is substituted in the integrals (7;5) a set of functions $I_p(b, \ell)$ is obtained having continuous first and second derivatives for values (b, ℓ) near those on E. The functional determinant $|\partial I_p/\partial \ell_q|$ $(p, q = 2, \ldots, n)$ is different from zero when $(b, \ell) = (b_0, \ell_0)$, as will be seen in the next paragraph. The equations $I_p(b, \ell) = c_p$ are satisfied by the values $(b, \ell) = (b_0, \ell_0)$, and hence have unique solutions $\ell_p(b, c)$ with $\ell_p(b_0, c_0) = \ell_{p0}$ and having continuous second derivatives near $(b, c) = (b_0, c_0)$. When the functions $\ell_p(b, c)$ are substituted for ℓ_p in the functions (7;6) a family (7;4) is obtained having the properties described in the lemma.

To complete the proof of Lemma 7:5 we have yet to show that the determinant $|\partial I_p/\partial \ell_q|$ is different from zero at $(b, \ell) = (b_0, \ell_0)$. First, we notice that no variation of the form $\eta = Y_{\ell_p}(x, b_0, \ell_0)a_p$ with the set $(a_p) \neq (0)$ is identically zero. For, if we substitute the functions (7:6) in the equation $dh_{y_0'}/dx - h_{y_0} = 0$, differentiate with respect to ℓ_p, multiply by a_p, and sum, we find that along E the equation

$$(7:7) \qquad d\omega_{\eta'}/dx - \omega_\eta + [dh_{py_0'}/dx - h_{py_0}]a_p = 0$$

holds, where ω is the function (4:7) and its arguments are $\eta = Y_{\ell_p}a_p$, $\eta' = Y_{\ell_p}'a_p$. If $\eta \equiv 0$, then equations (7:1) would hold with a set $(a) \neq (0)$, which is impossible according to Lemma 7:1. Next we notice that if $|\partial I_p/\partial \ell_q| = 0$, then there exist constants a_q not all zero such that the equations

$$(7:8) \qquad a_q\,\partial I_p/\partial \ell_q = \int_{x_1}^{x_2}(h_{py_0}\eta + h_{py_0'}\eta')dx = 0$$

hold, where $\eta = Y_{\ell_q}a_q$. Multiplying the left member of (7:7) by this η, integrating from x_1 to x_2 and using (7:8), we find that

$$(7:9) \qquad \int_{x_1}^{x_2}(\eta\, d\omega_{\eta'}/dx - \omega_\eta\eta)dx = 0.$$

With the usual integration by parts this becomes

$$(7:10) \qquad \sum_{r=1}^{w+1}\omega_{\eta'}\eta\Big|_{t_{r-1}}^{t_r} + \int_{x_1}^{x_2}2\omega\,dx = 0.$$

By differentiating with respect to ℓ_q the identities $Y(t_v, b, \ell) = b_v$ $(v = 0, 1, \ldots, w + 1)$, in which $b_0 = y_{01}$, $b_{w+1} = y_{02}$, we find that $Y_{\ell_q}(t_v, b, \ell) = 0$. Therefore $\eta(t_v) = Y_{\ell_q}(t_v, b_0, \ell_0)a_q = 0$, and (7:10) reduces to

(7:11) $$\int_{x_1}^{x_2} 2\omega dx = 0.$$

In view of equations (7:8) the functions

$$\eta_\beta(x) = \int_{x_1}^{x}(h_{\beta y_0}\eta + h_{\beta y_0'}\eta')dx \qquad (\beta = 1, \ldots, n)$$

satisfy the relations $\eta_1(x_1) = \eta_p(x_1) = \eta_p(x_2) = 0$ $(p = 2, \ldots, n)$. Hence $\eta_1(x_2) = 0$ also, by Lemma 7:2. It follows that the terms $g_{\beta\gamma}\eta_\beta(x_2)\eta_\gamma(x_2)$ outside the integral sign in the expression (4:8) for $J_2(\eta)$ vanish so that $J_2(\eta)$ reduces to the left member of (7:11). But then the equality (7:11) contradicts Theorem 6:2. Therefore our assumption that $|\partial I_p/\partial \ell_q| = 0$ is contrary to fact, and Lemma 7:5 is established.

We may now establish Theorem 7:1. First, let V and ϵ be chosen so that Theorem 6:1 holds, and let V_1, M, and L be chosen so that Lemma 7:4 holds. We diminish M and L, if necessary, so that the broken extremals E_ℓ of Lemma 7:4 satisfy the condition (6:1). That we may do this follows from the continuity properties of the family of broken extremals given in (7:4). We now let V_2 be a neighborhood of E in xy-space interior to V and V_1 and such that points $(x, y) = (t_s, b_s)$ of Lemma 7:4 lying in V_2 are also in M. Diminish ϵ if necessary so that the inequality $g_1(C) > 0$ holds for each curve C satisfying at $x = x_2$ the condition (6:1). This we may do since we have assumed that $g_1(E) > 0$. The region $V = V_2$ and the constant ϵ so chosen will be shown to have the property described in Theorem 7:1. To establish this we consider a given admissible arc C lying in V_2 joining the end points of E, and satisfying at $x = x_2$ the condition (6:1). By choosing the parameters b_s equal to the ordinates of C at t_s and the parameters c_p equal to the values of the integrals I_p evaluated along C we determine, as in Lemma 7:5, a broken extremal E_ℓ of the family

(7:4) passing through the points (t_s, b_s) and having $I_p(E_\ell) = I_p(C)$ $(p = 2, \ldots, n)$. By Lemma 7:4 we have

$$0 < J_\ell(C) - J_\ell(E_\ell) = \ell_\beta [I_\beta(C) - I_\beta(E_\ell)] = \ell_1 [I_1(C) - I_1(E_\ell)]$$

if $C \not\equiv E$. Since $\ell_1 = g_1(E) > 0$ it follows that $I_1(C) > I_1(E_\ell)$ unless $C \equiv E_\ell$. By Taylor's formula we have the relation

$$g(C) - g(E_\ell) = [I_1(C) - I_1(E_\ell)]g_1,$$

where the arguments of g_1 are $I_1(E_\ell) + \theta[I_1(C) - I_1(E_\ell)]$ $(0 < \theta < 1)$, and $I_p(E_\ell)$. But $g_1 > 0$ with these arguments by virtue of our choice of ϵ . Hence $g(C) > g(E_\ell)$ $(C \not\equiv E_\ell)$. By our choice of V_2 and ϵ the broken extremal E_ℓ satisfies the condition (6:1). We may then apply Theorem 6:1 to derive the result $g(E_\ell) \geqq g(E)$, the equality holding only if $E_\ell \equiv E$. It follows that $g(C) > g(E)$ $(C \not\equiv E)$. Theorem 7:1 has now been established.

8. *Application to a special problem*. In the introduction we noted that Bolza had treated in detail two problems discussed by Euler. Bolza generalized the problems formulated by Euler and by means of special methods gave sufficiency proofs for a strong relative minimum for one and maximum for the other. Both of the problems are problems of the type A formulated in Section 1. In this section we shall apply the theory developed for problem A to one of the problems treated by Bolza.

The first of the two problems considered by Bolza may be formulated as an instance of problem A as follows: In a class of admissible arcs joining the two fixed points $(x_1, y_1) = (-a, 0)$, $(x_2, y_2) = (a, 0)$ $(a > 0)$ we seek an arc minimizing the function

$$g(I_1, I_2) = I_2/(I_1 + b^2), \tag{8:1}$$

in which b^2 is a positive constant and I_1, I_2 are the area and length integrals

$$(8;2) \qquad I_1 = \int_{-a}^{a} y \, dx, \qquad I_2 = \int_{-a}^{a} \sqrt{1 + y'^2} \, dx.$$

This problem may be interpreted geometrically in the following manner: Let b^2 represent the area of the rectangle with corners at the points $(-a, 0)$, $(-a, -b^2/2a)$, $(a, -b^2/2a)$, $(a, 0)$. Then in a class of admissible arcs joining the two upper corners of the rectangle and satisfying the relation $y(x) > -b^2/2a$ $(-a < x < a)$ we seek an arc minimizing the ratio of the length of the arc to the area bounded by the arc and the base and sides of the above specified rectangle. We then take the region R of Section 1 to be the set of all points (x, y, y') having $y > -b^2/2a$.

On applying Theorem 2;1, that is, the multiplier rule, to the above special problem we find that a minimizing arc E for the problem must satisfy the equation

$$(8;3) \qquad \lambda_2 y'/\sqrt{1 + y'^2} = \lambda_1(x - x_1) + c,$$

in which λ_1, λ_2 are constant multipliers which satisfy with E the transversality condition

$$(8;4) \qquad \lambda_1[I_1 + b^2]^2 = -I_2, \qquad \lambda_2[I_1 + b^2] = 1.$$

We have set $\lambda_0 = 1$ in these equations. Solutions of (8;3) are functions defining arcs of circles and satisfying the equation

$$(8;5) \qquad (x + h)^2 + (y \pm k)^2 = (\lambda_2/\lambda_1)^2,$$

where h, k are constants. The branches of the circles (8;5) which pass through $(-a, 0)$, $(a, 0)$ and which are single valued on $-a \leqq x \leqq a$ are given by the formula

$$y = \pm(-k + \sqrt{R^2 - x^2}), \tag{8:6}$$

where $k > 0$ and $R = -\lambda_2/\lambda_1$. A minus sign appears in the last relation because $\lambda_2 > 0$, $\lambda_1 < 0$ for those of the arcs (8:6) which also satisfy equations (8:4). We calculate I_1, I_2 along the arcs (8:6) and then make the substitution $\omega = 2 \sin^{-1}(a/R)$. The result is

$$\begin{aligned} I_1 &= -a^2 \cot \omega/2 + a^2(\omega/2) \operatorname{cosec}^2 \omega/2, \\ I_2 &= a\, \omega \operatorname{cosec} \omega/2. \end{aligned} \tag{8:7}$$

By dividing the members of one of the equations (8:4) by the respective members of the other and then substituting the values (8:7) for I_1, I_2 and $a \operatorname{cosec} \omega/2$ for $(-\lambda_2/\lambda_1)$ one derives the equation (8:8) appearing in the following lemma:

LEMMA 8:1. *Circular arcs defined by the equation* (8:6), *joining the two fixed points* (-a, 0), (a, 0) *and subtended by a central angle* ω *satisfy the transversality condition if and only if* ω *is a solution of the equation*

$$b^2(1 - \cos\omega) - a^2(\omega + \sin\omega) = 0. \tag{8:8}$$

The equation (8:8) *has real roots when and only when* $b^2/a^2 \geqq \pi/2$. *If this relation holds without the equality sign one of the roots lies between* 0 *and* π, *the other between* π *and* 2π. *In this case let* E_0 *designate the arc of the family* (8:6) *whose central angle* ω *is the smaller of the two roots of* (8:8). *Then* E_0 *satisfies the condition* I *for our special problem.*

Bolza derived equation (8:8) by another method [II, pp. 249-251]. He established the readily-verified statement occurring in the lemma concerning the roots of the equation (8:8). We shall show in the succeeding paragraphs that besides satisfying the condition I, E_0 satisfies also the conditions II_N', III', IV' and the normality condition, so that we may apply Theorem 7:1 to establish a strong relative minimum for the function (8:1) along E_0. It will appear later that the arc determined by the larger of the roots of the equation (8:8) does not satisfy the relation $Q(z) > 0$. A similar statement applies when the two roots are equal, i.e., when $b^2/a^2 = \pi/2$. For that reason we limit our discussion to E_0.

The conditions II_N' and III' are strengthened forms of the conditions stated in Theorem 3:1. The function h appearing in Theorem 3:1 is for our special problem

$$h = \lambda_1 y + \lambda_2 \sqrt{1 + y'^2}. \tag{8:9}$$

Setting this value of h in the function $E(x, y_0, y_0', Y_0', \lambda)$ defined in (3:4) we find

$$\begin{aligned} E &= \lambda_2[\sqrt{1 + Y'^2} - \sqrt{1 + y'^2} - (Y' - y')y'/\sqrt{1 + y'^2}] \\ &= \lambda_2[\sqrt{(Y'y' + 1)^2 + (Y' - y')^2} - \sqrt{(Y'y' + 1)^2}]/\sqrt{1 + y'^2}. \end{aligned} \tag{8:10}$$

Examination of the second equation in (8:4) shows that $\lambda_2 > 0$. We readily see then that the condition II_N' is satisfied by E_0. Since the function $h_{y'y'}(x, y, y', \lambda)$ occurring in the condition III' is here

$$h_{y'y'} = \lambda_2/(1 + y'^2)^{3/2}, \tag{8:11}$$

we see that along E_0 the condition III' is also satisfied. We summarize the results of this paragraph in the following lemma:

LEMMA 8:2. The conditions II_N' and III' are satisfied along E_0. The Weierstrass E function for the special problem is given in (8:10), and the function appearing in the Clebsch condition is given in (8:11).

For our special problem the matrix (5:10) defining pairs of conjugate points contains 4 rows and at most 3 columns since for this problem $n = 2$ and the number of columns is $t + 2 \leqq n + 1$ by (5:7). Corollary 5:1 states that the functions appearing in the matrix (5:10) may be taken from the determinant (2:20), in which the elements are derivatives of the functions y_i, z_i taken with respect to parameters which are then evaluated on E_0. Accordingly, we differentiate the function $y = -k + \sqrt{R^2 - (x + h)^2}$, which is a part of the family (8:5) containing E_0, with respect to the constants k, h, R and set $h = 0$ to obtain the 3 functions

$$(8:12) \qquad \eta_{01} = -1, \qquad \eta_{02} = -x/\sqrt{R^2 - x^2}, \qquad \eta_{03} = R/\sqrt{R^2 - x^2}.$$

These 3 functions satisfy the relation $\eta_{0\pi}(x)a_\pi \not\equiv 0$ ($\pi = 1, 2, 3$) for all sets $(a) \neq (0)$. We have seen that there is a set of at most 3 such functions, so this set is a maximal set. The corresponding constants $\lambda_{\beta\pi}$ ($\beta = 1, 2$; $\pi = 1, 2, 3$), needed for the calculation of $Q(z)$, are found by differentiating $z_1 = -\ell_1$, $z_2 = \ell_1 R$ with respect to k, h, R and then setting the results equal to $\zeta_{\beta\pi} = -\lambda_{\beta\pi}$. In that manner we find

$$(8:13) \qquad \lambda_{1\pi} = \lambda_{21} = \lambda_{22} = 0, \qquad \lambda_{23} = -\ell_1.$$

To simplify further computations we make the substitution $\theta = \sin^{-1}x/R$. From (8:12) we have then

$$(8:14) \qquad \eta_{01} = -1, \qquad \eta_{02} = -\tan\theta, \qquad \eta_{03} = \sec\theta.$$

By use of equations similar to (5:8), in which $h_1 = y$, $h_2 = \sqrt{1 + y'^2}$, the functions $\eta_{\beta\pi}$ appearing in (5:10) are found to be

$$(8:15) \qquad \begin{aligned} \eta_{11} &= -R(\sin\theta - \sin\theta_1), & \eta_{21} &= 0, \\ \eta_{12} &= R(\cos\theta - \cos\theta_1), & \eta_{22} &= \sec\theta - \sec\theta_1, \\ \eta_{13} &= R(\theta - \theta_1), & \eta_{23} &= [\tan\theta_1 - \tan\theta + \theta - \theta_1], \end{aligned}$$

where $\theta_1 = -\sin^{-1}a/R$. The matrix (5:10) determining conjugate points is then

$$(8:16)\quad \left\Vert\begin{matrix} 1 & \tan\theta & \sec\theta \\ 1 & \tan\theta_1 & \sec\theta_1 \\ R(\sin\theta - \sin\theta_1) & R(\cos\theta_1 - \cos\theta) & R(\theta - \theta_1) \\ 0 & \sec\theta_1 - \sec\theta & \tan\theta_1 - \tan\theta + \theta - \theta_1 \end{matrix}\right\Vert$$

where the signs have been changed in the first two columns. In order to show that this matrix has rank 3 on $\theta_1 < \theta < \theta_2$, corresponding to $x_1 < x < x_2$, one may demonstrate that the determinant composed of the first, second and fourth rows has the value $2\sin(\theta - \theta_1)[\tan(\theta - \theta_1)/2 - (\theta - \theta_1)/2]/\cos\theta\cos\theta_1$. This expression does not vanish on the interval $\theta_1 < \theta < \theta_2 = -\theta_1$, provided $\theta_2 < \pi/2$. We note that $\omega/2 = \theta_2$ by definition, and that E_0 was determined by $\omega < \pi$. Therefore $\theta_2 < \pi/2$, and it follows that the matrix (8:16) has rank 3 on the interval $\theta_1 < \theta < \theta_2$. There is then no point 3 conjugate to 1 on $-a < x < a$, which is the interval corresponding to $\theta_1 < \theta < \theta_2$. We may note here that the range of θ is limited to the interval $-\pi/2$, $\pi/2$ by the requirement that admissible arcs be defined by single-valued functions $y(x)$. We have now established part of the following lemma:

LEMMA 8:3. *The arc* E_0 *satisfies the condition* IV'. *A matrix determining points conjugate to* 1 *is the matrix* (8:16) *which has rank* 3 *on* $\theta_1 < \theta < \theta_2$ *or* $x_1 < x < x_2$. *The function* $Q(z)$ *defined by* (5:7) *has the value*

$$(8:17)\qquad Q(z) = 2\ell_1(\theta_2 - \tan\theta_2)z^2$$

on E_0 *and satisfies the inequality* $Q(z) > 0$ *if* $z \neq 0$.

To complete the proof of this lemma we still must show that $Q(z) > 0$ holds on E_0. For the calculation of $Q(z)$ we need a linear combination of the set (8:14) which satisfies $\eta_0(\theta_1) = \eta_0(\theta_2) = 0$. Such a combination is $\eta_0 = \sec\theta - \sec\theta_2$. Corresponding linear combinations of the sets in (8:15) and (8:13) with $\theta_1 = -\theta_2$ yield

$$(8{:}18)\qquad \begin{aligned} \eta_1(x_2) &= 2R(\theta_2 - \tan\theta_2), & \lambda_1 &= 0, \\ \eta_2(x_2) &= 2(\theta_2 - \tan\theta_2), & \lambda_2 &= -\ell_1. \end{aligned}$$

For this set $Q(z)$ has the value

$$(8{:}19)\quad Q(z) = z^2[4(g_{11}R^2 + 2g_{12}R)(\theta_2 - \tan\theta_2)^2 + 2\ell_1(\theta_2 - \tan\theta_2)].$$

By use of the relations (8:7), (8:8) and $a \operatorname{cosec}\theta_2 = R$, $\omega = 2\theta_2$, one may show that along E_0

$$(8{:}20)\qquad I_1 + b^2 = 2\theta_2 R^2, \qquad I_2 = 2\theta_2 R.$$

These equations may then be combined with the equation (8:19) and the definition of g given in (8:1) to yield the value of $Q(z)$ appearing in (8:17). Since $\ell_1 < 0$, $Q(z) > 0$ holds provided $0 < \theta_2 < \pi/2$. This last relation implies $b^2/a^2 > \pi/2$, and conversely, as we noted earlier. Lemma 8:3 has now been established.

One readily verifies that the equations

$$dh_{1y'}/dx - h_{1y} = 0, \qquad dh_{2y'}/dx - h_{2y} = 0$$

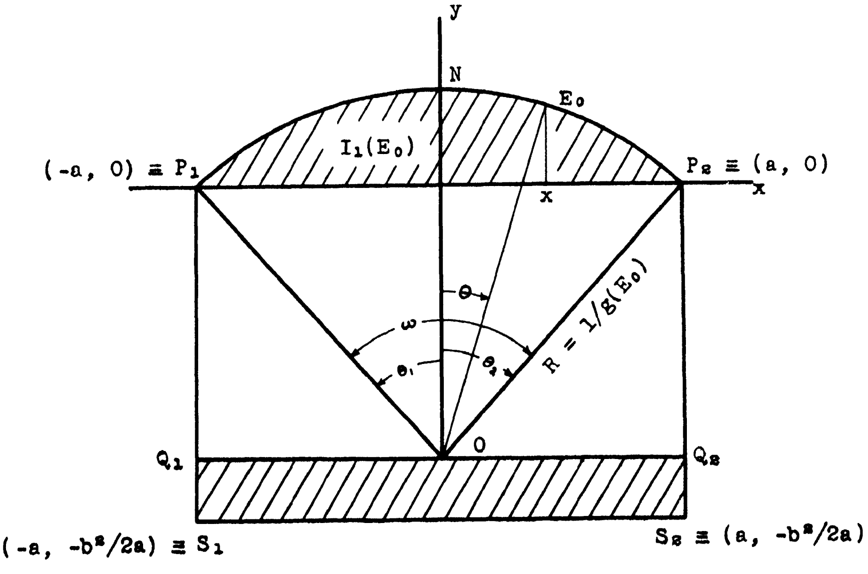
do not hold along E_0, and that therefore E_0 satisfies the normality condition defined in Section 7. By virtue of this statement and Lemmas 8:1, 8:2 and 8:3, the arc E_0 satisfies all the hypotheses of Theorem 7:1, and the following theorem is proved:

THEOREM 8:1. *For the special problem considered in this section there exists an extremal* E_0 *defined as in Lemma* 8:1 *such* *that the function* g *defined in* (8:1) *takes on a strong relative minimum on* E_0 *provided only that* $b^2/a^2 > \pi/2$.

The transversality condition for the special problem may be interpreted geometrically. Euler [I, p. 154] showed that the minimum value of g is equal to the curvature of the minimizing arc. We may verify this by dividing the members of the second of equations (8:20) by the respective members of the first of the equations. That division yields $g(E_0) = 1/R$, where R is the radius of the circle of which E_0 is an arc. The first of the equations (8:20) further shows that on E_0

(8:21)
$$b^2 = 2\theta_2 R^2 - I_1 = \omega R^2 - I_1,$$

or the constant b^2 is equal to twice the area of the sector subtended by ω less the area I_1 between the arc E_0 and the chord (-a, 0), (a, 0). Euler also expressed this fact, though in a somewhat different manner. The above relations and the relations between ω, θ, and x are shown in the figure which follows.



If we represent an area by a succession of letters designating its perimeter then by (8:21) we have

$$\begin{aligned} b^2 &= 2OP_1NP_2 - I_1(E_0) = 2OP_1P_2 + I_1(E_0) \\ &= P_1Q_1Q_2P_2 + I_1(E_0). \end{aligned}$$

But $b^2 = P_1S_1S_2P_2$ so that $I_1(E_0) = Q_1S_1S_2Q_2$. Thus we see that the center O of the minimizing arc E_0 must lie far enough above S_1S_2 to make the area of the rectangle $Q_1S_1S_2Q_2$ lying immediately below O equal to the area $I_1(E_0)$. A consequence of this statement is that the y coordinate of O must satisfy the relation $y > -b^2/2a$.

9. A second application. By replacing the function g in (8:1) by the function

$$g(I_1, I_2) = I_2[I_1 + b^2] \tag{9:1}$$

we have a statement in the first part of Section 8 of the second problem considered by Bolza. The discussion of this second problem may be made to follow the discussion given in Section 8 for the first problem with a few changes of formulas and results. We shall note briefly most of the necessary changes. The transversality condition (8:4) should be replaced by $\ell_1 = I_2$, $\ell_2 = I_1 + b^2$. This condition leads to the equation

$$b^2(1 - \cos\omega) + a^2(3\omega - \sin\omega) = 0 \tag{9:2}$$

rather than (8:8). This equation has distinct real roots only when $b^2/a^2 > 1/\chi$, where

$$\chi = \frac{\sin 2\gamma}{3 - \cos 2\gamma} = .2785 \text{ approx.,} \tag{9:3}$$

the angle γ occurring here being the root of

$$3\gamma = 2 \tan \gamma \qquad (9:4)$$

lying between 0 and $\pi/2$. Approximately $\gamma = 55^\circ\ 26'$. The equations (9:2), (9:3) and (9:4) occur in Bolza's treatment of this problem [II, pp. 254-255]. In Lemma 8:1 we determine E_0 for the second problem by the root of (9:2) lying between 0 and -2γ. Then $\theta_2 = \omega/2$ lies between 0 and $-\gamma$. With this value of θ_2 determining the range of θ, $\theta_1 \geqq \theta \geqq \theta_2 = -\theta_1$, one may readily verify as before that the rank of the matrix (8:16) is 3 on $\theta_2\theta_1$, or that E_0 satisfies the conjugate point condition. Instead of (8:19) one has

$$Q(z) = z^2[8R(\theta_2 - \tan \theta_2)^2 + 2\ell_1(\theta_2 - \tan \theta_2)].$$

By use of the equations

$$I_1 + b^2 = -2\theta_2 R^2 = \ell_2, \qquad I_2 = 2\theta_2 R = \ell_1, \qquad (9:5)$$

which correspond to the equations (8:20), one may reduce $Q(z)$ to

$$Q(z) = 4Rz^2(\theta_2 - \tan \theta_2)(3\theta_2 - 2 \tan \theta_2), \qquad (9:6)$$

corresponding to (8:17). The values $R = a \operatorname{cosec} \omega/2$ and θ_2 occurring here are negative. The relation $Q(z) > 0$ then holds for $-\gamma < \theta_2 < 0$, where γ has the value defined above. We may note that $Q(z) = 0$ when $\theta_2 = -\gamma$. The Weierstrass and Clebsch conditions of Lemma 8:2 are satisfied along E_0 since $\ell_2 > 0$, as is shown by the equations (9:5). Corresponding to Theorem 8:1 we have the following theorem:

THEOREM 9:1. *For the second problem considered by Bolza and discussed above there exists an extremal* E_0 *such that the function* g *defined in* (9:1) *takes on a strong relative minimum on* E_0, *provided only that* $b^2/a^2 > 1/\chi$, *where* χ *has the value given in* (9:3).

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THE PROBLEM OF BOLZA FOR DOUBLE INTEGRALS
IN THE
CALCULUS OF VARIATIONS

BY

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THE PROBLEM OF BOLZA FOR DOUBLE INTEGRALS
IN THE CALCULUS OF VARIATIONS

1. Introduction. The problem of the calculus of variations discussed in this paper is an extension to double integrals of the problem of Bolza for simple integrals [4][1]. Closely related questions have been considered by Hilbert [9] and more recently by Tucker [14], who has used the methods of tensor analysis. Stated briefly, the problem is that of finding in a class of admissible surfaces $z = z(x,y)$, whose boundary curves are required to lie on a fixed capstan-shaped surface, one which minimizes a sum of the form

$$I = \iint_A f(x,y,z,p,q)dx\,dy + \int_L g(x,y,z,x',y',z')ds.$$

The double integral is taken over a region A which is the projection of a portion of an admissible surface on the xy-plane and the line integral is evaluated in the positive sense along the boundary curve L of that surface. This problem is also a generalization of a problem discussed by Bolza [5] and by Simmons [12] of minimizing a double integral with variable limits. In Section 2, definitions and hypotheses sufficient for an analysis of the problem are given. These permit a more precise statement of the problem than that given above. A coordinate system which is of service throughout the paper is defined in Section 3. By means of a slight modification of the method used by Simmons for differentiating a double integral with variable limits, a differentiation formula for integrals is obtained. This formula, stated in Section 4, is useful in deriving the first and second variations. Section 5 is devoted to the construction of a family of admissible surfaces embedding a given surface and having prescribed variations along the boundary curve. A converse problem is also treated. The first variation and an associated transversality condition are presented in Section 6. The transversality condition is obtained without the assumption of non-tangency made by Simmons. Of special interest is the calculation of a simple form for the second variation, as shown in Section 7. In Section 8 a Jacobi condition phrased in terms of a boundary value problem is stated. Analogues of the necessary conditions of Weierstrass and Legendre which must hold along the boundary curve of a minimizing surface are given in Section 9. In the final section an application of the results of this paper is made to an illustrative example.

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[1] The numbers in square brackets refer to the bibliography at the end of the paper.

2. Analytic basis. In this section definitions and hypotheses sufficient for an analysis of the problem proposed in the introduction are given, and the problem is then restated.

An arc in xyz-space which is continuous and which consists of a finite number of sub-arcs each of which has a continuously turning tangent, will be termed a regular arc. This implies that on each sub-arc the parameter may be so chosen that the functions $x(u)$, $y(u)$, $z(u)$ defining the arc have continuous derivatives such that $x'^2 + y'^2 + z'^2$ is different from zero. An analogous definition holds for a regular arc in the plane.

A regular surface $z = z(x,y)$, with (x,y) on a region A of the xy-plane, is one for which the arc bounding the region A is regular, and for which the function $z(x,y)$ is single valued and is piecewise of class C', that is, it is continuous and has continuous first partial derivatives in a neighborhood of the region A; or it is a continuous surface consisting of a finite number of partial surfaces each of which has these properties.[1]

It is assumed that there is a five-dimensional open region R_1 of points (x,y,z,p,q) such that the function f is of class C''' in R_1.

It is further assumed that there is a six-dimensional open region R_2 of points (x,y,z,x',y',z') in which the following conditions are satisfied: (a) if the set (x',y',z') is in R_2 and k is positive, then the set (kx',ky',kz') is in R_2; (b) the function g is of class C''' and is positively homogeneous of order one in (x',y',z') for (x,y,z,x',y',z') in R_2. In our subsequent discussion, a set (x,y,z,x',y',z') will frequently be denoted more simply by (x,x').

By an admissible surface we shall mean a regular surface defined by an equation of the form

$$(2:1) \qquad z = z(x,y) \qquad (x,y) \text{ on } A$$

whose elements (x,y,z,z_x,z_y) are interior to R_1, and whose boundary curve L is a simply closed regular arc with elements (x,x') interior to R_2. It follows that no two points of L can lie on the same parallel to the z-axis. The orthogonal projection of the curve L on the xy-plane is assumed to be simply closed regular arc C bounding the region A. The curve L is said to be described in the positive sense when the curve C is described in the positive sense.

---

[1] A function is said to be of class $C^{(n)}$ in a region R if it is continuous in R and all of its derivatives of orders 1, 2, ... , n are continuous in R.

A fixed capstan-shaped surface may be represented in parametric form by the equations

$$(2{:}2) \qquad x = X(u,v), \quad y = Y(u,v), \quad z = Z(u,v),$$

where $0 \leqq u \leqq \lambda$, $-\epsilon \leqq v \leqq +\epsilon$. The functions $X(u,v)$, $Y(u,v)$, $Z(u,v)$ are assumed to be of class C''' in their arguments, and to have a positive real period λ in the variable u for values v on the interval $-\epsilon \leqq v \leqq +\epsilon$. It is also assumed that the values (x,y,z) defined by the equations (2:2) are interior to the region R_3 and that the fixed surface has no singular points for values of u and v under consideration.

The problem of Bolza for double integrals considered in this paper is now that of finding the class of admissible surfaces with equations of the type (2:1) whose boundary curves are required to lie on the fixed surface (2:2), one which gives the functional

$$I = \iint_A f(x,y,z,p,q)\,dx\,dy + \int_L g(x,x')\,ds$$

a minimum value. Here $p = \partial z/\partial x$, $q = \partial z/\partial y$ and the line integral is taken in the positive sense along the boundary curve L of an admissible surface S whose projection on the xy-plane is the region A. The arguments of the function g are those belonging to the curve L, and accents denote derivatives taken with respect to the parameter s, which represents arc length on the boundary curve. As a consequence of the homogeneity condition imposed on the function g, the value of the line integral is independent of the parametric representation of the boundary curve, provided the positive sense on the curve L is preserved. If the region R_2 of points (x,x') described above is such that the set $(x,-x')$ is in R_2 whenever (x,x') is in R_2 and if the relation $g(x,-x') = g(x,x')$ holds for every set (x,x') in R_2, then the value of the line integral appearing in the sum I is independent of the orientation of the boundary curve L of the surface S. This is the so-called reversible case.

Throughout this paper we shall be concerned with a surface S_0 of class C'' which we assume minimizes the functional I. Our problem is then that of finding the special properties which a minimizing surface of this type must possess. It will be convenient to assume that the boundary curve L_0 of S_0 is given by equations (2:2) with $v = 0$. Such a representation of the capstan surface may always be obtained provided the normal lines to this surface along the curve L_0 are nowhere parallel to the xy-plane. A simply closed curve L may be defined on the capstan surface by specifying a function $v = v(u)$ which is of class C'' and has a period χ in the variable u. The functional to be minimized then has the form

$$(2:3) \qquad I = \iint_A f(x,y,z,p,q)\,dx\,dy + \int_0^{\chi} g(x,x')\,du,$$

where the arguments of g are those belonging to the curve L, accents denote derivatives taken with respect to the variable u, and the region A is defined as before.

3. A coordinate system. Let C_0 be a simply closed curve in the xy-plane defined by equations of the form

$$x = x_0(u), \qquad y = y_0(u) \qquad (0 \leqq u \leqq \ell),$$

where the functions $x_0(u)$, $y_0(u)$ are of class C''', have a period ℓ in the variable u, and have $x_0'^2 + y_0'^2 \neq 0$ along C_0. If the parameter u represents arc length on C_0, then ℓ is the length of C_0, and $x_0'^2 + y_0'^2 = 1$. We assume for definiteness that the parameter u measures in the positive direction on C_0. In a neighborhood of C_0 a uv-coordinate transformation may be defined by the equations

$$(3{:}1) \qquad x = X(u,v), \qquad y = Y(u,v) \qquad (-\epsilon \leqq v \leqq +\epsilon).$$

The functions $X(u,v)$, $Y(u,v)$ have the following properties: (1) they are of class C''' in their arguments, (2) they have a period ℓ in the variable u, (3) the equations $x_0(u) = X(u,0)$, $y_0(u) = Y(u,0)$ are identities in u, (4) the functional determinant

$$(3{:}2) \qquad D(u,v) = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix}$$

is different from zero along C_0; that is, for $v = 0$. For example, we might take the curves v = constant as the parallel curves to C_0 and the curves u = constant as the normals to C_0. In Simmons' paper [12, p. 237], the positive direction of the variable v was taken along the outer normal to C_0, and hence the uv-directions were oriented oppositely to that of the xy-directions. The functional determinant (3:2) had the value $D = -(1 + v/\rho)$, where ρ is the radius of curvature of C_0, and on C_0, $D = -1$. In this paper, however, the positive direction of the variable v is taken to be directed interior to the region bounded by C_0, resulting in a similarly directed orientation of the xy-directions and the uv-directions. The determinant D in equation (3:2) is therefore positive for values of v sufficiently near zero.

Let $v = v(u,a)$ be a function of class C'' for all sets (u,a) with $0 \leqq u \leqq \lambda$ and the parameter a sufficiently near zero, with the further properties that $v(u,a)$ has the period λ in the variable u, and $v(u,0) \equiv 0$. It follows that the equations

$$(3:3) \qquad x = X[u,v(u,a)], \qquad y = Y[u,v(u,a)]$$

define a one parameter family of simply closed curves C_a in the xy-plane for the parameter a near zero. The family contains the curve C_0 for the parameter value $a = 0$, and each curve of the family has the period λ in the variable u. Denote by A_a the area bounded by the curve C_a. The determinant $D[u,v(u,a)]$ obtained by setting $v = v(u,a)$ in equation (3:2) remains positive when the parameter a is sufficiently near zero.

4. A differentiation formula. Before deriving the first and second variations for the functional I defined in Section 2, it is convenient to develop a method for differentiating a sum of the type (2:3) which involves a parameter in the limits of integration of the double integral and in both integrand functions. The procedure employed is a simple modification of a method given by Simmons [12, p. 238].

Let C^* be a simply closed curve defined by the equation (3:1) for a constant positive value $v = v^*$ near zero, and let A^* be the region bounded by C^*. Denote by ΔA the area included between the curves C^* and C_a. Consider the expression

$$(4:1) \qquad J(a) = \iint_{A_a} F(x,y,a)\,dx\,dy + \int_0^{\lambda} G(u,a)\,du,$$

where $F(x,y,a)$ is of class C'' for all sets (x,y,a) with (x,y) in a neighborhood of the area A_0 bounded by C_0 and having the parameter a sufficiently near zero; and where $G(u,a)$ is of class C'' for all sets (u,a) with $0 \leqq u \leqq \chi$ and the parameter a near zero. The single integral is to be evaluated along the curve C_a. We may then write

$$J(a) = \iint_{A^*} F\,dx\,dy + \iint_{\Delta A} F\,dx\,dy + \int_0^{\chi} G\,du,$$

where in the first double integral the parameter a occurs only in the integrand. It is only the second double integral whose differentiation presents difficulties. This double integral may be transformed to uv-coordinates by means of equations (3:1) and has the value:

$$(4:2) \qquad \int_0^{\chi}\int_{v(u,a)}^{v^*} F[X(u,v),\ Y(u,v),\ a]D(u,v)\,dv\,du.$$

The derivative of the expression (4:2) is found by the usual formula for differentiating a definite integral with respect to a parameter occurring in the integrand and in the limits of integration to be

$$(4:3) \qquad \int_0^{\chi}\int_{v(u,a)}^{v^*} \frac{\partial F}{\partial a} D\,dv\,du - \int_0^{\chi} F\,D\,v_a\,du,$$

where in the single integral the variable v, wherever it occurs, is the function $v(u,a)$. If the double integral appearing in the expression (4:3) is expressed in terms of xy-coordinates, a formula for $J'(a)$ is obtained.

LEMMA 4:1. The first derivative of the expression $J(a)$ defined in equation (4:1) has the value

$$(4:4) \qquad J'(a) = \iint_{A_a} \frac{\partial F}{\partial a}\,dx\,dy + \int_0^{\chi} \left(\frac{\partial G}{\partial a} - F\,D\,v_a\right)du.$$

An explicit expression for $J''(a)$ may similarly be calculated, but will not be used in deriving the second variation for the functional I defined by equation (2:3), since it will be shown that it is much simpler to calculate the second variation of I after the transversality condition associated with the first variation has been established.

5. *Embedding theorems.* When a function $v(u,a)$ having the properties described in Section 3 is substituted for v in the equation (2:2) of the fixed surface, a one parameter family of simply closed curves L_a is obtained with the equations

$$(5:1) \qquad x = X[u,v(u,a)], \qquad y = Y[u,v(u,a)], \qquad z = Z[u,v(u,a)].$$

This family contains a curve L_0 for $a = 0$. The variations of the family along L_0 have the form

$$(5:2) \qquad \xi_1 = X_v v_a, \qquad \xi_2 = Y_v v_a, \qquad \xi_3 = Z_v v_a,$$

where the arguments of X_v, Y_v, Z_v, v_a are $(u,0)$. The functions $\xi_1(u)$, $\xi_2(u)$, $\xi_3(u)$ have the period λ in the variable u. Let S_0 be a surface of class C'' with an equation of the form $z = z(x,y)$, with (x,y) in A, and having the curve L_0 as its boundary. In this section we first establish the following theorem:

THEOREM 5:1. *Let* $v(u,a)$ *be a function of class* C'' *for all sets* (u,a) *with* $0 \leqq u \leqq \lambda$ *and* a *near zero such that* $v(0,a) = v(\lambda,a)$ *and* $v(u,0) \equiv 0$. *Then there exists a one parameter family of admissible surfaces* $z = z(x,y,a)$, *defined and piecewise of class* C'' *for the parameter* a *near zero, which contains the given surface* S_0 *for* $a = 0$, *intersects the fixed surface* (2:2) *in the curves of the family* (5:1) *for corresponding values of the parameter, and has the prescribed variations* (5:2) *along the boundary curve* L_0 *on which* $a = 0$.

In order to prove this theorem, let v^* be a fixed positive value of v, sufficiently near zero, so the uv-coordinate system is well defined. A fixed simply closed curve L^* lying on the given surface S_0 may be determined by the equations

$$x = X(u,v^*), \qquad y = Y(u,v^*), \qquad z = z[X(u,v^*), Y(u,v^*)].$$

For a fixed value of the parameter a, we consider a surface having curvilinear coordinates u and w and defined by the equations

$$\begin{aligned} x &= (1 - w)X(u,v) + wX(u,v^*), \\ y &= (1 - w)Y(u,v) + wY(u,v^*), \\ z &= (1 - w)Z(u,v) + wz[X(u,v^*), Y(u,v^*)], \end{aligned} \tag{5:3}$$

where $v = v(u,a)$ and $0 \leqq w \leqq 1$. The boundary of this surface is composed of the curves L_a and L^*, as is seen by setting $w = 0$ and $w = 1$, respectively. When a value $u = u_0$ on the interval $(0,\ell)$ is selected, the equations (5:3) reduce to the equations of a straight line connecting the points on the curves L_a and L^* for which $u = u_0$.

The surface (5:3) together with the portion of S_0 bounded by the curve L^* defines a surface S_a whose boundary is L_a. It remains to show that the surface S_a is expressible in the form $z = z(x,y,a)$ and is admissible. To do this, it is sufficient to show that the first two equations in (5:3) may be solved for the variables u and w as functions of x, y, a. The functional determinant of these two equations formed with respect to the variables u and w has the value

$$\Delta(u,w,v^*\,a) = \begin{vmatrix} (1 - w)(X_u + X_v v_u) + wX_u{}^*, & X^* - X \\ \\ (1 - w)(Y_u + Y_v v_u) + wY_u{}^*, & Y^* - Y \end{vmatrix}$$

where the arguments of the unstarred terms in X, Y and their derivatives are u, $v(u,a)$ and of the starred terms are u, v^*. We may regard v^* as a parameter and obtain by the theorem of mean value the expressions

$$X(u,v^*) - X(u,v) = (v^* - v)X_v[u,\ v + \theta(v^* - v)],$$
$$Y(u,v^*) - Y(u,v) = (v^* - v)Y_v[u,\ v + \theta(v^* - v)],$$

where $v = v(u,a)$ and $0 \leqq \theta \leqq 1$. If these expressions are substituted into the second column of the determinant Δ, we obtain a relation of the form

$$(5{:}4) \qquad \Delta(u,w,v^*,a) = (v^* - v)D(u,w,v^*,a)$$

where

$$D(u,w,v^*,a) = \begin{vmatrix} (1 - w)(X_u + X_v v_u) + wX_u^*, & X_v^* \\ (1 - w)(Y_u + Y_v v_u) + wY_u^*, & Y_v^* \end{vmatrix},$$

and where X_v^* and Y_v^* have the arguments u, $v + \theta(v^* - v)$. When $v(u,0) \equiv 0$, we have $v_u(u,0) \equiv 0$ and the determinant $D(u,w,0,0)$ is identical with the determinant (3:2) with $v = 0$ and is therefore different from zero. It follows that there is a constant v^* near zero and a constant $d > 0$ such that the expressions $D(u,w,v^*,a)$ and $v^* - v(u,a)$ are different from zero for $0 \leqq u \leqq \ell$, $0 \leqq w \leqq 1$, $-d \leqq a \leqq d$. From the formula (5:4) it is seen that $\Delta(u,w,v^*,a)$ is different from zero for these values of v^* and a. By the use of implicit function theorems the first two equations of (5:3) have solutions

$$u = U(x,y,a), \qquad w = W(x,y,a),$$

which are of class C'' in (x,y,a) for the parameter a sufficiently near zero and for values (x,y) in a neighborhood of those belonging to L_0. If we substitute these functions into the third equation of (5:3), then for a fixed value of a an admissible surface $z = \Psi(x,y,a)$ of class C'' is determined which intersects the fixed surface in the curve L_a. The surface S_a is therefore representable in the form $z = z(x,y,a)$ and is piecewise of class C'' and admissible. As the parameter a varies, a one parameter family of surfaces of the above type is obtained which contains the given surface S_0 for $a = 0$ and has the desired variations (5:2) along L_0. This completes the proof of Theorem 5:1.

We next prove the following theorem, which is a converse of the preceding:

THEOREM 5:2. *Let* $z = z(x,y,a)$ *be a one parameter family of admissible surfaces piecewise of class* C'' *for the parameter* a *near zero, and containing a surface* S_0 *for* $a = 0$ *which is nowhere tangent to the fixed surface* (2:2) *along their intersection. Then the intersection of this family with the fixed surface determines a one parameter family of simply closed curves* L_a *which is defined by a function* $v(u,a)$ *of class* C'' *which has the period* χ *in the variable* u *and has* $v(u,0) \equiv 0$.

In particular, the family $z = z(x,y,a)$ may be chosen in the form $z = z(x,y) + a\zeta(x,y)$, where $z = z(x,y)$ is the equation for S_0 and $\zeta(x,y)$ is a function piecewise of class C''.

To prove the theorem, we note that the equation

$$z[X(u,v),\, Y(u,v), a] - Z(u,v) = 0 \tag{5:5}$$

holds for values $0 \leqq u \leqq \chi$ when $v = 0$, $a = 0$, since the boundary curve L_0 of S_0 is given by equations (2:2) with $v = 0$. The derivatives of the left member of this equation with respect to the variables u and v are the expressions

$$\rho(u,v,a) = pX_u + qY_u - Z_u, \tag{5:6}$$

$$\sigma(u,v,a) = pX_v + qY_v - Z_v. \tag{5:7}$$

Since the surface S_0 is by hypothesis nowhere tangent to the surface (2:2) along the curve L_0, the quantities $\rho(u,0,0)$, $\sigma(u,0,0)$ cannot vanish simultaneously at any point on L_0. Since $\rho(u,0,0) \equiv 0$, it follows that $\sigma(u,0,0) \neq 0$. By the extended implicit function theorem [2], the equation (5:5) has a solution $v = v(u,a)$ of class C'' on $0 \leqq u \leqq \lambda$, $|a| < a_0$. As a result of the initial conditions, one has $v(u,0) \equiv 0$. Furthermore, since the functions $X(u,v)$, $Y(u,v)$, $Z(u,v)$ have a period λ in the variable u, the function $v(u,a)$ also has this periodicity property. It is to be noted that the expression $\sigma(u,v(u,a),a)$ is different from zero when the parameter a is sufficiently small.

6. *The first variation.* In order to obtain the first variation of the functional I to be minimized, we suppose that a one parameter family of admissible surfaces S_a of the type

$$z = z(x,y,a) \tag{6:1}$$

has been constructed containing a surface S_0 for $a = 0$, and is defined for the parameter a near zero. This family intersects the fixed surface (2:2) in a family of simply closed curves L_a with the equations

$$x = X[u,\ v(u,a)], \qquad y = Y[u,\ v(u,a)], \qquad z = Z[u,\ v(u,a)], \tag{6:2}$$

where the function $v(u,a)$ has the properties described in Section 3. The possibility of the construction of the families (6:1) and (6:2) has been discussed in Theorems 5:1 and 5:2 of the preceding section.

We further assume that $z_a(x,y,a)$ is piecewise of class C'' for the parameter a near zero.

Before deriving the expression for the first variation, we develop a number of useful formulas. If we substitute the function $v = v(u,a)$ into equation (5:5) and differentiate the resulting identity with respect to the variables a and u, we obtain the important relations

$$(6{:}3) \qquad z_a = -\sigma v_a, \qquad\qquad \rho = -\sigma v_u,$$

where the expressions for ρ and σ are given in equations (5:6) and (5:7). At $a = 0$ the relations $v_u = 0$, $\rho = 0$ are satisfied. It is to be noted from the first equation in (6:3) that the variations z_a and v_a are not both arbitrary when $\sigma \neq 0$.

The direction parameters of the normal line to the fixed surface (2:2) at any point are proportional to the determinants

$$D_1 = \begin{vmatrix} Y_u & Z_u \\ Y_v & Z_v \end{vmatrix}, \qquad D_2 = \begin{vmatrix} Z_u & X_u \\ Z_v & X_v \end{vmatrix}, \qquad D = \begin{vmatrix} X_u & Y_u \\ X_v & Y_v \end{vmatrix},$$

where the arguments of the derivatives of X, Y, Z are (u,v). Since by hypothesis $D(u,v)$ is different from zero when $v = 0$, we may apply implicit function theorems and solve the first two equations in (2:2) for u and v as functions of x and y. The fixed surface (2:2) may therefore be represented in the non-parametric form $\Phi(x,y,z) = 0$ in a neighborhood of the curve on which $v = 0$. Since $\Phi(X,Y,Z) = 0$ is an identity in u and v we obtain by differentiation the orthogonality relations

$$\Phi_x X_u + \Phi_y Y_u + \Phi_z Z_u = 0,$$
$$\Phi_x X_v + \Phi_y Y_v + \Phi_z Z_v = 0.$$

If these equations are solved for Φ_x and Φ_y, we find the relations $D\Phi_x = D_1\Phi_z$, $D\Phi_y = D_2\Phi_z$. If Φ_z were zero, then Φ_x and Φ_y are also zero, contrary to the assumption that the fixed surface is free of singular points. If we set $p_1 = -\Phi_x/\Phi_z$, $q_1 = -\Phi_y/\Phi_z$ we obtain the relations

$$-Dp_1 = D_1, \qquad -Dq_1 = D_2, \tag{6:4}$$

where $(p_1, q_1, -1)$ are now direction parameters for the normal line to the fixed surface (2:2) at any point. It is important to note that these equations are valid when v is replaced by $v(u,a)$, provided the parameter a is sufficiently small. If Z_u and Z_v are eliminated from the determinants D_1 and D_2 in equations (6:4) by means of the relations (5:6) and (5:7) for ρ and σ, we obtain the equations

$$\begin{aligned} \sigma X_u - \rho X_v &= DQ, \\ \sigma Y_u - \rho Y_v &= -DP, \end{aligned} \tag{6:5}$$

where $P = p - p_1$, $Q = q - q_1$. We may now write

$$\rho = PX_u + QY_u, \qquad \sigma = PX_v + QY_v. \tag{6:6}$$

Since $\rho = -\sigma v_u$, equations (6:5) may be written in the useful form

$$\begin{aligned} \sigma X' &= \sigma(X_u + X_v v_u) = DQ, \\ \sigma Y' &= \sigma(Y_u + Y_v v_u) = -DP, \end{aligned} \tag{6:7}$$

where accents denote derivatives taken with respect to the variable u. When $a = 0$, equations (6:7) reduce to $\sigma X_u = DQ$, $\sigma Y_u = -DP$.

If the sum I defined in equation (2:3) is evaluated for a surface of the family (6:1), an expression of the form studied in

Section 4 is obtained, since the limits of integration of the single integral in equation (2:3) may be chosen to be independent of the parameter a. On applying the result stated in equation (4:4), we find the expression

$$(6:8) \qquad I'(a) = \iint_{A_a} \frac{\partial f}{\partial a}\, dx\, dy + \int_0^{\lambda} \left(\frac{\partial g}{\partial a} - fDv_a\right) du,$$

where A_a is the area bounded by the projection on the xy-plane of the curve of intersection L_a of the fixed surface with that surface of the family (6:1) whose parameter value is a. The arguments of the derivatives of f and g are, respectively, those belonging to the surface S_a and its boundary curve L_a. Throughout the single integral the variable v, wherever it occurs, is the function v(u,a). The Euler expression has the definition

$$(6:9) \qquad R(x,y,a) = f_z - \frac{\partial}{\partial x} f_p - \frac{\partial}{\partial y} f_q,$$

where the arguments of the derivatives of f are those belonging to the surface S_a. It follows from the theory of the problem of minimizing a double integral with a fixed boundary curve [3, p.5] that R vanishes at a = 0 when S_0 is a minimizing surface. It is of course necessary that this condition hold along a minimizing surface for the problem stated in Section 2. By means of the relation

$$\frac{\partial f}{\partial a} = z_a R + \frac{\partial}{\partial x}(f_p z_a) + \frac{\partial}{\partial y}(f_q z_a)$$

and an application of Green's Theorem, the double integral in equation (6:8) has the value

$$\iint_{A_a} z_a R\, dx\, dy + \int_0^{\lambda} z_a (f_p Y' - f_q X')\, du,$$

provided the surface S_a is piecewise of class C'' for values (x,y) in A_a. We may write

$$\frac{\partial g}{\partial a} = \sum (g_x x_a + g_{x'} x_a'),$$

where the sum indicated represents a sum of three terms obtained by replacing the letter x by x, y, z in turn. From the relations $x_a = X_v v_a$, $x_a' = X_v v_a' + X_v' v_a$ and similar ones for y and z, it follows that $\partial g/\partial a$ may be written in the form

$$\frac{\partial g}{\partial a} = v_a \sum (g_x - \frac{d}{du} g_{x'}) X_v + \frac{d}{du}(v_a \sum g_{x'} X_v).$$

The last term of this expression vanishes when integrated between the limits 0 and λ, since there are no corners on the boundary curve and the functions occurring all have the period λ in the variable u. If we collect the terms of the expression (6:8), use $z_a = -\sigma v_a$ and the relations (6:7), we obtain the following result:

THEOREM 6:1. The first derivative of the expression $I(a)$, when taken over that portion of the surface S_a which is bounded by the fixed surface (2:2), has the value

$$I'(a) = \iint_{A_a} z_a R \, dx \, dy + \int_0^{\lambda} v_a T \, du, \tag{6:10}$$

where R is defined in equation (6:9) and $T(u,a)$ is the expression

$$T = D(Pf_p + Qf_q - f) + \sum (g_x - \frac{d}{du} g_{x'}) X_v, \tag{6:11}$$

and $P = p - p_1$, $Q = q - q_1$.

If S_0 is a minimizing surface of class C'' for the functional I, then a first necessary condition is $I'(0) = 0$, and as previously mentioned, $R = 0$. Since by Theorem 5:1 the variation v_a is an arbitrary function of class C'', it follows from the fundamental lemma for simple integrals that $T = 0$ along the boundary curve L_0 of S_0. Hence we have

THEOREM 6:2. *At each point on the boundary curve* L_0 *of a surface* S_0 *of class* C'' *which minimizes the functional* I, *the transversality condition* $T = 0$ *must be satisfied, where* T *is defined by equation* (6:11) *with* $a = 0$.

If the fixed capstan surface (2:2) is represented in the non-parametric form $\Phi(x,y,z) = 0$, then by means of the relations $p_1 = -\Phi_x/\Phi_z$, $q_1 = -\Phi_y/\Phi_z$, the transversality condition may be written in the form

$$D[f_p\Phi_x + f_q\Phi_y + (pf_p + qf_q - f)\Phi_z] + \Phi_z \Sigma \left(g_x - \frac{d}{du} g_{x'}\right)X_v = 0.$$

If the function g is identically zero, the non-zero factor D may be deleted and the transversality condition reduces to that given by Simmons [12, p. 245]. An alternative form for the transversality condition in this case is $f = (p - p_1)f_p + (q - q_1)f_q$. It follows that the minimizing surface cannot be tangent to the capstan surface if the function f is different from zero along L_0. This remark is justifiable since an examination of the derivation of the transversality condition $T = 0$ shows that no use has been made of the non-tangency assumption stated in Theorem 5:2.

As a consequence of the homogeneity condition imposed on the function g, the identity

$$g(x,x') = \Sigma\, x' g_{x'}(x,x')$$

holds. If this identity is differentiated with respect to u and the set (x,x') is replaced by the set $[X(u,0),\ X_u(u,0)]$, we obtain the relation $\Sigma\, \lambda X_u = \lambda X_u + \mu Y_u + \nu Z_u = 0$, where

$$(6:12) \qquad \lambda = g_x - \frac{d}{du} g_{x'}, \qquad \mu = g_y - \frac{d}{du} g_{y'}, \qquad \nu = g_z - \frac{d}{du} g_{z'}$$

are abbreviations for the Euler directions. If the function f is identically zero, then the transversality condition reduces to $\sum \lambda X_v = 0$. From the relations $\sum \lambda X_u = 0$, $\sum \lambda X_v = 0$, it is seen that the Euler directions λ, μ, ν coincide with the direction of the normal line to the fixed surface (2:2), unless they vanish simultaneously. At any point on the boundary curve L_0, the direction parameters of the normal line to the fixed surface, considered in the non-parametric form $\Phi(x,y,z) = 0$, are proportional to the derivatives Φ_x, Φ_y, Φ_z. There then exists a proportionality factor $\tau(u)$ such that the relations

$$(6:13) \qquad \lambda = \tau \Phi_x, \qquad \mu = \tau \Phi_y, \qquad \nu = \tau \Phi_z$$

hold along L_0. Define $G(x,x') = g(x,x') - \tau \Phi$. It is readily verified that the equations (6:13) can be written in the form

$$(6:14) \quad G_x - \frac{d}{du} G_{x'} = 0, \qquad G_y - \frac{d}{du} G_{y'} = 0, \qquad G_z - \frac{d}{du} G_{z'} = 0.$$

This proves the following result:

THEOREM 6:3. *The Euler equations* (6:14) *for the problem of minimizing the functional* I *defined by equation* (2:3) *when* f *is identically zero, follow as a consequence of the transversality condition described in Theorem* 6:2.

7. *The second variation.* In this section we make the non-tangency assumption stated in Theorem 5:2. The differentiation formula of Section 4 may be applied to the expression for the first variation given in equation (6:10). We then obtain

$$I''(a) = \iint_{A_a} (z_{aa}R + z_a \frac{\partial R}{\partial a}) dx\, dy + \int_0^{\chi} (v_{aa}T + v_a \frac{\partial T}{\partial a} - z_a R D v_a) du.$$

If S_0 is a minimizing surface for the functional I, then for $a = 0$ the relations $R = 0$, $T = 0$, $I''(0) \geq 0$ must be satisfied. The above equation therefore reduces to

$$(7:1) \qquad I''(0) = \iint_{A_0} z_a \frac{\partial R}{\partial a}\, dx\, dy + \int_0^{\ell} v_a \frac{\partial T}{\partial a}\, du.$$

Let $2\omega(x,y,\zeta,\pi,\kappa)$ be the quadratic form

$$(7:2) \quad 2\omega = f_{zz}\zeta^2 + f_{pp}\pi^2 + f_{qq}\kappa^2 + 2(f_{zp}\zeta\pi + f_{pq}\pi\kappa + f_{qz}\kappa\zeta),$$

where $\zeta(x,y) = z_a(x,y,0)$, $\pi(x,y) = \zeta_x$, $\kappa(x,y) = \zeta_y$. By differentiation the Euler expression (6:9) with respect to the parameter a it is readily established that for a = 0 the Jacobi expression

$$(7:3) \qquad J = \omega_\zeta - \frac{\partial}{\partial x}\omega_\pi - \frac{\partial}{\partial y}\omega_\kappa$$

is equal to $\partial R/\partial a$. As a consequence of the homogeneity property

$$2\omega = \zeta\omega_\zeta + \pi\omega_\pi + \kappa\omega_\kappa ,$$

the functions J and 2ω are related by the equation

$$(7:4) \qquad \zeta J = 2\omega - \frac{\partial}{\partial x}(\zeta\omega_\pi) - \frac{\partial}{\partial y}(\zeta\omega_\kappa).$$

If this value for ζJ is substituted into equation (7:1), then by an application of Green's Theorem and by making use of the first equation in (6:3) and the equation (6:7), the expression for $I''(0)$ may be put in the form

$$(7:5) \quad I''(0) = \iint_{A_0} 2\omega\, dx\, dy + \int_0^{\ell} v_a\left[\frac{\partial T}{\partial a} - D(P\omega_\pi + Q\omega_\kappa)\right] du.$$

The equation (6:11) may be written $T(u,a) = DA + B$, where

$$(7:6) \qquad A = Pf_p + Qf_q - f, \qquad B = \Sigma\lambda X_v,$$

and λ, μ, ν are defined in equations (6:12). If we differentiate equation (6:11) with respect to a, we find

$$\frac{\partial T}{\partial a} = (D_v A + DA_v)v_a + DA_a + \frac{\partial B}{\partial a}. \tag{7:7}$$

Here A_a represents the derivative of A with respect to the parameter a wherever it occurs explicitly, and has the value

$$A_a = P\omega_\pi + Q\omega_\kappa - f_z\zeta .$$

This may be shown by differentiating the first equation in (7:6) with respect to a and using the relations $f_{pa} = \omega_\pi$, $f_{qa} = \omega_\kappa$, $f_a = f_z\zeta + f_p\pi + f_q\kappa$, $P_a = p_a = \pi$, $Q_a = q_a = \kappa$. The derivatives ω_π and ω_κ are readily computed from equation (7:2). If we use the relation $v_a = -\zeta/\sigma$, where $\sigma = PX_v + QY_v$, the expression (7:5) for $I''(0)$ becomes

$$I''(0) = \iint_{A_0} 2\omega\, dx\, dy + \int_0^{\chi} \frac{\zeta^2}{\sigma^2}[D_v A + D(A_v + \sigma f_z)]\, du + \int_0^{\chi} v_a \frac{\partial B}{\partial a}\, du. \tag{7:8}$$

We proceed to evaluate the last integral in (7:8). From the second equation in (7:6) we obtain

$$\frac{\partial B}{\partial a} = v_a \Sigma \lambda X_{vv} + \Sigma \lambda_a X_v .$$

The expression $\Sigma \lambda_a X_v$ may be written as

$$\Sigma \lambda_a X_v = \Sigma X_v \frac{\partial g_x}{\partial a} + \Sigma X_v' \frac{\partial g_{x'}}{\partial a} - \frac{d}{du}\Sigma X_v \frac{\partial g_{x'}}{\partial a} .$$

If we use this relation, then the term $v_a(\partial B/\partial a)$ occurring in the equation (7:8) for the second variation has the value

$$\begin{aligned} v_a \frac{\partial B}{\partial a} = {} & v_a^2 \Sigma \lambda X_{vv} + \Sigma X_v v_a \frac{\partial g_x}{\partial a} + \Sigma (X_v' v_a + X_v v_a')\frac{\partial g_{x'}}{\partial a} \\ & - \frac{d}{du}\left(v_a \Sigma X_v \frac{\partial g_{x'}}{\partial a}\right). \end{aligned} \tag{7:9}$$

The last expression in the right member of this equation vanishes when integrated between the limits 0 and χ, since each term has the period ℓ in the variable u. The variations $\xi_1 = X_v v_a$, $\xi_2 = Y_v v_a$, $\xi_3 = Z_v v_a$ have the derivatives

$$\xi_1' = X_v' v_a + X_v v_a', \quad \xi_2' = Y_v' v_a + Y_v v_a', \quad \xi_3' = Z_v' v_a + Z_v v_a'.$$

Let $2\Omega(\xi, \xi')$ represent a quadratic form in the variables $(\xi, \xi') = (\xi_1, \xi_2, \xi_3, \xi_1', \xi_2', \xi_3')$ whose matrix of coefficients is of the sixth order and consists of the second derivatives of the function $g(x,x')$ with respect to the variables x,y,z,x',y',z'. By using equation (7:9), the relation $v_a = -\zeta/\sigma$ and the definition $2\Omega(\xi, \xi')$, then the equation (7:8) for the second variation can be put into the form

$$\begin{aligned} I''(0) = \iint_{A_0} 2\omega \, dx \, dy + \int_0^{\ell} \frac{\zeta^2}{\sigma^2}[D_v A + D(A_v + \sigma f_z) + \Sigma \lambda X_{vv}] du \\ + \int_0^{\ell} 2\Omega(\xi, \xi') du, \end{aligned} \tag{7:10}$$

where 2ω is defined by equation (7:2) and 2Ω is the quadratic form described above.

We note that the double integral occurring in equation (7:10) is the same as that for the double integral problem with fixed limits of integration. In the first simple integral the integrand has the factor ζ^2 but contains no other powers or derivatives of the variation ζ. If we use $\sigma = PX_v + QY_v$, then the expression $A_v + \sigma f_z$ occurring in $I''(0)$ has the value

$$A_v + \sigma f_z = (A_x + Pf_z)X_v + (A_y + Qf_z)Y_v,$$

where explicit expressions for A_x and A_y are

$$A_x = P_x f_p + Q_x f_q + P(f_p)_x + Q(f_q)_x - (f)_x,$$
$$A_y = P_y f_p + Q_y f_q + P(f_p)_y + Q(f_q)_y - (f)_y,$$

The notation $(f)_x$, for example, is an abbreviation for the total derivative of f with respect to x, and has the value

$$(f)_x = f_x + f_z p + f_p r + f_q s,$$

where $r = p_x = z_{xx}$, $s = p_y = q_x = z_{xy}$, $t = q_y = z_{yy}$. If r_1, s_1, t_1 represent the corresponding second derivatives for the fixed surface (2:2), considered in non-parametric form, then

$$P_x = r - r_1, \qquad P_y = s - s_1 = Q_x, \qquad Q_y = t - t_1.$$

It may be verified by using the Euler equation $f_z = (f_p)_x + (f_q)_y$ and the relation $Q_x = P_y$, that we may write

$$\begin{aligned} A_x + Pf_z &= P(f_p)_x + Q(f_q)_x - (f)_x + (Pf_p)_x + (Pf_q)_y, \\ A_y + Qf_z &= P(f_p)_y + Q(f_q)_y - (f)_y + (Qf_p)_x + (Qf_q)_y. \end{aligned} \tag{7:11}$$

With the aid of equation (3:2) defining D, we find

$$D_v = X_u Y_{vv} - Y_u X_{vv} + Y_v X_{uv} - X_v Y_{uv}.$$

Another expression for D_v may be obtained if we specialize the coordinate system described in Section 3. We have a normal coordinate system, in the sense that the variable v measures along the inner normal to C_0, if we impose the restrictions $DX_v = -Y_u$, $DY_v = X_u$. By means of these relations and the equation (3:2) for D the equations

$$D^2 = X_u^2 + Y_u^2, \qquad X_v^2 + Y_v^2 = 1, \qquad X_u X_v + Y_u Y_v = 0$$

are readily established. From the equations (6:7) with $v = v_u = 0$ and the above equations we obtain the additional relations

$$\sigma X_v = P, \qquad \sigma Y_v = Q, \qquad \sigma^2 = P^2 + Q^2, \tag{7:12}$$

which hold when $v = 0$. An expression for D_v may be calculated at once by differentiating $D^2 = X_u^2 + Y_u^2$ with respect to v.

The derivatives X_{uv}, Y_{uv} may be found from the relations $X_v = P/\sigma$, $Y_v = Q/\sigma$. We then find, on using $\rho = PX_u + QY_u = 0$, $X_u = DQ/\sigma$, $Y_u = -DP/\sigma$ and the relations for P_x, P_y, Q_x, Q_y the expression

$$D_v = (D/\sigma^3)[Q^2(r - r_1) - 2PQ(s - s_1) + P^2(t - t_1)].$$

Define $G(x,x') = g(x,x') - \tau(u)\Phi(x,y,z)$, where $\tau(u)$ is an arbitrary function of u of class C'' with the period γ in the variable u and $\Phi(x,y,z) = 0$ is the equation of the capstan surface (2:2) considered in non-parametric form. If we define

$$\lambda^* = G_x - \frac{d}{du}G_{x'}, \quad \mu^* = G_y - \frac{d}{du}G_{y'}, \quad \nu^* = G_z - \frac{d}{du}G_{z'},$$

then the relations $\lambda^* = \lambda - \tau\Phi_x$, $\mu^* = \mu - \tau\Phi_y$, $\nu^* = \nu - \tau\Phi_z$ are readily established. The expression $(\zeta^2/\sigma^2)\sum\lambda X_{vv}$ occurring in equation (7:10) has the value

$$v_a^2\sum\lambda X_{vv} = v_a^2[\sum\lambda^* X_{vv} + \tau\sum\Phi_x X_{vv}].$$

From the identity $\sum\Phi_x X_v = 0$, we obtain by differentiation the relation $\sum\Phi_x X_{vv} = -\sum\Phi_{xv}X_v$, where

$$\Phi_{xv} = \Phi_{xx}X_v + \Phi_{xy}Y_v + \Phi_{xz}Z_v,$$

and similar expressions hold for Φ_{yv} and Φ_{zv}. Let $2\Omega^*(\xi, \xi')$ be a quadratic form in the variables (ξ, ξ') whose coefficients are the second derivatives of the function $G(x,x')$ with respect to the variables x,y,z,x',y',z'. If we use $v_a = -\zeta/\sigma$, then equation (7:10) may be put in the form indicated in the following theorem:

THEOREM 7:1. _The second derivative of the integral sum_ I(a), _evaluated at_ a = 0 _for a minimizing surface_ S_0 _whose projection on the_ xy-_plane is the region_ A_0, _has the value_

$$I''(0) = \iint_{A_0} 2\omega\, dx\, dy + \int_0^{\chi} \frac{\zeta^2}{\sigma^2}[D_v A + D(A_v + \sigma f_z) + \Sigma \lambda^* X_{vv}]\, du$$

$$(7{:}13) \qquad\qquad + \int_0^{\chi} 2\Omega^*(\xi, \xi')\, du,$$

where $2\Omega^*(\xi, \xi')$ *is the quadratic form described above.*

When the function f is identically zero, it is seen by equation (6:14) that there exists a function $\tau(u)$ such that λ^*, μ^*, ν^* all vanish. The expression (7:13) for the second variation then reduces to

$$I''(0) = \int_0^{\chi} 2\Omega^*(\xi, \xi')\, du.$$

If the function g is identically zero, then $B \equiv 0$ and the transversality condition stated in Theorem 6:2 reduces to $A = 0$, where A is defined in the first of equations (7:6). If we make use of the normal coordinate system described above, then by equations (7:12) the expression (7:13) for the second variation becomes

$$I''(0) = \iint_{A_0} 2\omega\, dx\, dy + \int_0^{\chi} \frac{\zeta^2 D}{\sigma^3}[P(A_x + Pf_z) + Q(A_y + Qf_z)]\, du,$$

where $A_x + Pf_z$ and $A_y + Qf_z$ have the values given in equations (7:11). If we set $D = 1$, this result for $I''(0)$ agrees with the equations (49) and (50) defining $I''(0)$ as given in a paper by Simmons [12, p. 248], after a minor correction of equation (50) is made. The sum $P^2(f_p)_x + Q^2(f_q)_y$ which occurs twice in the expression A_2 of equation (50) of that paper should occur only once. It should be noted in correlating the work of this paper with that of Simmons that he made use of a coordinate transformation for which $D = -1$ when $v = 0$.

8. A boundary value problem. A Jacobi condition phrased in terms of a boundary value problem associated with the second variation may be formulated in a manner analogous to that used by Simmons [12, p. 249]. We first express $I''(0)$ in a form involving the Jacobi expression J defined in equation (7:3). To do this, we substitute the value of 2ω given by equation (7:4) into the equation (7:13) with $\tau(u) = 0$ and apply Green's Theorem. By using the equations (6:7) with $a = 0$ we obtain the expression

$$I''(0) = \iint_{A_0} \zeta J \, dx \, dy - \int_0^{\chi} \frac{\zeta}{\sigma} [D(P \omega_\pi + Q \omega_\kappa) - \zeta H(u)] \, du + \int_0^{\chi} 2\Omega(\xi, \xi') \, du, \tag{8:1}$$

where

$$H(u) = \frac{1}{\sigma}[D_v A + D(A_v + \sigma f_z) + \sum \lambda X_{vv}].$$

We proceed to express the last written integral in another form. By means of the homogeneity property

$$2\Omega(\xi, \xi') = \sum \xi_i \Omega_{\xi_i} + \sum \xi_i' \Omega_{\xi_i'},$$

where the sums are taken for $i = 1, 2, 3$, we obtain

$$2\Omega(\xi, \xi') = \sum \xi_i (\Omega_{\xi_i} - \frac{d}{du} \Omega_{\xi_i'}) + \frac{d}{du}(\sum \xi_i \Omega_{\xi_i'}). \tag{8:2}$$

The last expression in the right member of (8:2) vanishes when integrated between the limits 0 and χ, since each term has the period χ in the variable u. By using the relation $v_a = -\zeta/\sigma$, the variations ξ_1, ξ_2, ξ_3 have the values $\xi_1 = -\zeta X_v/\sigma$, $\xi_2 = -\zeta Y_v/\sigma$, $\xi_3 = -\zeta Z_v/\sigma$. With the aid of these relations and the values of ω_π, ω_κ as computed from equation (7:2), the expression (8:1) may be written as

$$\text{(8:3)}\qquad I''(0) = \iint_{A_0} \zeta J \, dx \, dy - \int_0^{\ell} \frac{\zeta}{\sigma} \Big[L\zeta + M\pi + N\kappa + \Sigma \Big(\Omega_{\xi_1} - \frac{d}{du} \Omega_{\xi_1'} \Big) X_v \Big] du,$$

Where the functions L, M, N are defined by the equations

$$\text{(8:4)}\qquad \begin{aligned} L &= D(Pf_{pz} + Qf_{qz}) - H(u) \\ M &= D(Pf_{pp} + Qf_{qp}) \\ N &= D(Pf_{pq} + Qf_{qq}), \end{aligned}$$

and $\pi = \zeta_x$, $\kappa = \zeta_y$, $P = p - p_1$, $Q = q - q_1$.

The boundary value problem may now be stated in the following form:

THEOREM 8:1. In order that the surface $z = z(x,y)$ shall minimize the functional I defined in equation (2:3), it is necessary that for negative values of λ^* the boundary value problem

$$\text{(8:5)}\qquad \begin{aligned} & J(\zeta) - \lambda^*\zeta = 0 && \underline{\text{in}}\ A_0, \\ & L\zeta + M\pi + N\kappa + \Sigma \Big(\Omega_{\xi_1} - \frac{d}{du} \Omega_{\xi_1'} \Big) X_v = 0 && \underline{\text{on}}\ C_0, \end{aligned}$$

have no solution except $\zeta \equiv 0$, where C_0 is the boundary of A_0. The functions L, M, N are defined in equations (8:4).

The proof that the boundary value problem has no solution except $\zeta \equiv 0$ is made by observing that the line integral in equation (8:3) is zero for every admissible function $\zeta(x,y)$ satisfying the second equation in (8:5), and that a function ζ which satisfies both equations gives $I''(0)$ the value

$$I''(0) = \lambda^* \iint_{A_0} \zeta^2 dx \, dy.$$

If ζ were not identically zero, then for $\lambda^* < 0$ the second variation would be negative and the integral sum I would not be minimized.

9. The necessary conditions of Weierstrass and Legendre. For the problem of minimizing a double integral in a class of surfaces having a common simply closed boundary curve, the necessary conditions of Weierstrass and Legendre are well known [3, pp. 9, 27]. These conditions must also hold along a minimizing surface for the problem of this paper. In addition, it will be shown that along the boundary curve of the minimizing surface a Weierstrass condition formed for the function g must be satisfied. For a problem closely related to that of this paper, Hilbert [9] has pointed out the fact that two Weierstrass conditions play a role; one for the interior of a minimizing surface and one for its boundary.

We shall employ a slight modification of the method used by Graves [8] for establishing the Weierstrass condition for multiple integral variation problems. The procedure consists in setting up two one parameter families of admissible surfaces defined over a small portion of the minimizing surface and a segment of its boundary. When the parameter, which occurs in each of the families, is allowed to approach zero, the entire configuration shrinks to a point on the boundary curve in a manner to be described below.

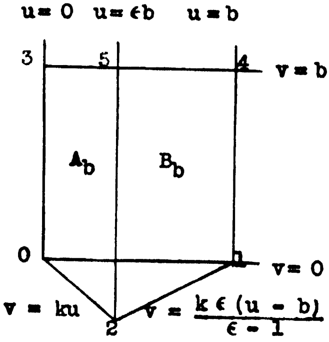

FIGURE 1

We consider a rectangular coordinate system in the uv-plane. A square may be defined in this plane bounded by the lines $u = 0$, $u = b$, $v = 0$, $v = b$, where b is a parameter $\geqq 0$. Lines passing through the points 0 and 1 respectively, as shown in Figure 1, are defined by the equations $v = ku$, $v = k\epsilon(u - b)/(\epsilon - 1)$, where $0 < \epsilon < 1$ and k is different from zero. These lines intersect on the line $u = \epsilon b$ at a point 2. In Figure 1, we have for convenience regarded k as negative. Let A_b and B_b denote respectively the areas of the trapezoidal regions 0352 and 1452. Then the values

$$(9:1) \qquad A_b = b^2\epsilon(1 - k\epsilon/2), \quad B_b = b^2(1 - \epsilon)(1 - k\epsilon/2)$$

are easily computed.

The expression for the functional I defined in equation (2:3) may be transformed to uv-coordinates by means of the equations (3:1), which are now assumed to define a normal coordinate system, in the sense described in the last part of Section 7. We consider a portion S_{ob} of the minimizing surface S_o whose projection on the uv-plane is the region 0143, as shown in Figure 1. Let the remaining part of S_o be denoted by S_o^*, and let L_o^* be that part of its boundary which is on the fixed surface. For the portion S_{ob} of the minimizing surface we may write

$$I(b) = \int_0^b \int_0^b f\, D\, dv\, du + \int_0^{\cancel{l}} g\, du,$$

where D is defined by equation (3:2) and b is positive and sufficiently near zero, so the coordinate transformation (3:1) is well defined. The arguments of the function g are those belonging to a segment L_{ob} of the boundary curve L_o of the minimizing surface whose projection on the uv-plane is the line 01. The curve L_{ob} lies on the fixed surface (2:2) and is determined by values $0 \leqq u \leqq b$, $v = 0$. The equations $v = ku$, $v = k\epsilon(u - b)/(\epsilon - 1)$ also define curves on the fixed surface. These will be denoted by L_{1b} and L_{2b} respectively. It will be shown later in this section that for each fixed value of the parameter b near zero, an admissible comparison surface S may be constructed which is composed of S_0^* and two surfaces S_{1b} and S_{2b}. The surfaces S_{1b} and S_{2b} form a cap over the portion S_{0b} of the minimizing surface and have as their projections on the uv-plane the regions A_b and B_b respectively. The boundary curve of the comparison surface S is a continuous curve lying on the fixed surface, and consists of the curves L_0^*, L_{1b}, L_{2b}. Define

$$J(b) = \iint_{A_b} fD\,dv\,du + \iint_{B_b} fD\,dv\,du + \int_0^{\epsilon b} g\,du + \int_{\epsilon b}^{b} g\,du.$$

Here the arguments of the integrand function f in the first double integral are those belonging to the surface S_{1b} and in the second double integral are those belonging to the surface S_{2b}. The first single integral is to be evaluated along the curve L_{1b} and the second along the curve L_{2b}.

Since the surface S_{0b} is part of a minimizing surface for the integral sum defined in equation (2:3), we obtain, after eliminating the value of this sum over the surface S_0^* and the boundary curve L_0^*, the inequality

$$J(b) - I(b) \geqq 0, \tag{9:2}$$

which must be satisfied for values of the parameter b sufficiently near zero. This inequality may be written as $K_1(b) + K_2(b) \geqq 0$, where

$$K_1(b) = \iint_{A_b} fD\,dv\,du + \iint_{B_b} fD\,dv\,du - \int_0^b\int_0^b fD\,dv\,du,$$

$$K_2(b) = \int_0^{\epsilon b} g\,du + \int_{\epsilon b}^{b} g\,du - \int_0^b g\,du,$$

and the arguments of the integrand functions in the expressions for $K_1(b)$ and $K_2(b)$ are as described above. If we apply the theorem of mean value for double integrals to $K_1(b)$, then by using equations (9:1) we find an expression of the form $K_1(b) = b^2H(b)$, where $H(b)$ is a function which is bounded for the parameter b sufficiently small, since the functions f and D are bounded in a neighborhood of the values $u = 0$, $v = 0$. The function g may be regarded as a function of the arguments u, v, v'. By an application of the theorem of mean value for single integrals to $K_2(b)$, the inequality (9:2) becomes

$$b^2H(b) + \epsilon bg(u_1,ku_1,k) + b(1-\epsilon)g\left[u_2, \frac{k\epsilon(u_2 - b)}{\epsilon - 1}, \frac{k\epsilon}{\epsilon - 1}\right]$$

$$- bg(u_3,0,0) \geqq 0,$$

where $0 < u_1 < \epsilon b$, $\epsilon b < u_2 < b$, $0 < u_3 < b$. If we divide this inequality by b, let b approach zero, and introduce the original arguments $(X, X_u + X_v v')$ for g, we obtain a form of the Weierstrass condition due to Graves [8]:

$$\epsilon\,[g(X,X_u + kX_v) - g(X,X_u)]$$

$$+ (1-\epsilon)[g(X,X_u + \frac{k\epsilon}{\epsilon - 1}X_v) - g(X,X_u)] \geqq 0,$$

where the arguments (u,v) are the set (0,0). One of the usual forms for the Weierstrass condition is obtained if we divide this inequality by ϵ, set $h = \epsilon/(\epsilon - 1)$ and let ϵ approach zero. The E-function has a variety of representations. In our present notation one form is the expression

$$(9{:}3) \qquad E(X,X_u,X') = g(X,X') - g(X,X_u) - \Sigma\,(X' - X_u)g_{x'}(X,X_u),$$

where $X' = X_u + kX_v$, $Y' = Y_u + kY_v$, $Z' = Z_u + kZ_v$, $k \neq 0$. As a consequence of the homogeneity condition imposed on the function g, the E-function may be written in the simpler form

$$E(X,X_u,X') = g(X,X') - \sum X' g_{X'}(X,X_u).$$

At any point P_0 on the boundary curve of a minimizing surface, we consider the lines having the direction parameters (X_u, Y_u, Z_u) and (X_v,Y_v,Z_v) and their projections on the xy-plane. The arguments of the derivatives of X, Y, Z are the set (u,0). A line with the direction parameters $(X_v,Y_v,0)$ separates the xy-plane into two parts. Let π be the half-plane in the positive direction of a line with the direction parameters $(X_u, Y_u, 0)$. Then if a line having the direction parameters (X',Y',Z') passes through the point P_0, lies in the tangent plane to the fixed surface (2:2) at this point, and has as its projection on the xy-plane a line directed interior to the half-plane π , we call (X',Y',Z') an *admissible direction*. If in place of the positive homogeneity assumption postulated for the function g in Section 2, it is assumed that the stronger homogeneity property $g(x,-x') = g(x,x')$ is satisfied, then every line through the point P_0 and lying in the tangent plane to the fixed surface is said to have an admissible direction. Except for the problem of constructing an admissible comparison surface S, we have justified the following result:

THEOREM 9:1. *Let L_0 be the boundary curve of a surface minimizing the functional I defined in equation (2:3). Then the Weierstrass condition*

$$E(X, X_u,X') \geqq 0$$

must be satisfied at every element (X,X_u) of L_0 for all admissible directions (X',Y',Z') distinct from (X_u,Y_u,Z_u).

We now consider the problem of constructing the comparison surface S discussed earlier in this section. Two regions, denoted by A_b and B_b, have been defined in the uv-plane. In a neighborhood of $v = 0$ the minimizing surface S_0 has in uv-coordinates the equation

$$z = z[X(u,v),\ Y(u,v)] \equiv z^*(u,v) \qquad (0 \leqq u \leqq l).$$

We are considering a portion S_{0b} of S_0 defined by the inequalities $0 \leqq u \leqq b$, $0 \leqq v \leqq b$. Along the segment L_{0b} of the boundary of this surface the incident relation $z^*(u,0) = Z(u,0)$ is satisfied. We regard the parameters k, ϵ, b as fixed, and may define two admissible surfaces S_{1b} and S_{2b} which have as their projections on the uv-plane the regions A_b and B_b respectively. We define the surface S_{1b} by the equation $z = z_1(u,v)$, where for $k < 0$

$$\begin{aligned} z_1(u,v) &= z^*(u,v) \quad \text{when } 0 \leqq v \leqq b, \\ &= Z(u,v) \quad \text{when } ku \leqq v \leqq 0, \end{aligned}$$

and where the variable u has the range $0 \leqq u \leqq \epsilon b$. When $k > 0$, we may define

$$z_1(u,v) = z^*(u,v) + \frac{b - v}{b - ku}\,[Z(u,ku) - z^*(u,ku)],$$

where $0 \leqq u \leqq \epsilon b$, $ku \leqq v \leqq b$, $\epsilon k < 1$. Let $\tau = k\epsilon(u - b)/(\epsilon - 1)$ Then the surface S_{2b} may be defined by the equation $z = z_2(u,v)$, where for $k < 0$

$$\begin{aligned} z_2(u,v) &= z^*(u,v) \quad \text{when } 0 \leqq v \leqq b, \\ &= Z(u,v) \quad \text{when } \tau \leqq v \leqq 0, \end{aligned}$$

and the variable u has the range $\epsilon b \leqq u \leqq b$. When $k > 0$, we may define

$$z_2(u,v) = z^*(u,v) + \frac{b - v}{b - \tau}\,[Z(u, \tau) - z^*(u, \tau)],$$

where $\epsilon b \leqq u \leqq b$, $\tau \leqq v \leqq b$, $\epsilon k < 1$. It may be verified that the comparison surface S thus defined has the properties required to carry through the proof of the Weierstrass condition.

We may now establish a Legendre condition as a consequence of the Weierstrass condition stated in Theorem 9:1. The E-function defined in equation (9:3), when considered as a function of k, has a minimum when $k = 0$, and it follows that the relations $E'(0) = 0$, $E''(0) \geqq 0$ must be satisfied. The first relation is found to be satisfied identically, and the second relation gives a form of the Legendre condition.

THEOREM 9:2. *Let* L_0 *be the boundary curve of a surface minimizing the functional* I *defined in equation* (2:3). *Then at each element* (X, X_u) *of the boundary curve the Legendre condition*

$$\begin{aligned} &X_v^2 g_{x'x'} + Y_v^2 g_{y'y'} + Z_v^2 g_{z'z'} \\ (9{:}4) \qquad &+ 2(X_vY_vg_{x'y'} + Y_vZ_vg_{y'z'} + Z_vX_vg_{z'x'}) \geqq 0 \end{aligned}$$

must hold for all directions (X_v, Y_v, Z_v) *distinct from* (X_u, Y_u, Z_u) *and lying in the tangent plane to the fixed surface* (2:2) *at any point on* L_0. *The arguments of the second derivatives of the function* g *are the set* (X, X_u) *defining the element on the boundary.*

If the fixed surface (2:2) is represented in the non-parametric form $\Phi(x,y,z) = 0$, then Theorem 9:2 may be stated in a slightly different form. It is readily shown that the inequality (9:4) must hold for all directions (X_v, Y_v, Z_v) which are distinct from (X_u, Y_u, Z_u) and which satisfy the orthogonality relation $\sum \Phi_x X_v = 0$ along L_0.

If $2\psi(\xi_i;\eta_i)$ represents a bilinear form in the variables ξ_1, ξ_2, ξ_3 and η_1, η_2, η_3 whose coefficients are the second derivatives of g with respect to x', y', z', then (a) $2\psi(X_u;X_u) = 0$, (b) $2\psi(X_u;X_v) = 0$, and (c) $2\psi(X_v;X_v) \geqq 0$. The result (a) is a consequence of the homogeneity condition imposed on the function g. It may be obtained by differentiating the identity $g(X,X_u) = \Sigma X_u g_{x'}(X,X_u)$ with respect to X_u, Y_u, Z_u in turn, then multiplying the resulting three equations by X_u, Y_u, Z_u respectively, and adding. A computation similar to that used to derive (a) establishes the relation (b). The relation (c) is merely another notation for the inequality (9:4). This inequality may now be written in the form $2\psi(X';X') \geqq 0$, as is seen from the equation

$$2\psi(X';X') = 2\psi(X_u;X_u) + 2k\cdot 2\psi(X_u;X_v) + k^2\cdot 2\psi(X_v;X_v)$$

and the relations (a), (b), (c) above. Hence it follows that if the Legendre condition (9:4) holds for one direction, it holds for all directions of the type indicated.

10. *An illustrative example*. In this section an application of the results of this paper will be made to the problem of minimizing the expression I defined in equation (2:3) when the functions f and g have the special values

$$f = \sqrt{1 + p^2 + q^2}, \qquad g = \sqrt{x'^2 + y'^2 + z'^2}. \tag{10:1}$$

We may assume that the fixed surface is so parametrized that u represents arc length along the boundary curve L_0 of a minimizing surface, and hence $g^2 = x'^2 + y'^2 + z'^2 = 1$ on L_0. We then find the derivatives $f_p = p/f$, $f_q = q/f$, $\lambda = -x''$, $\mu = -y''$, $\nu = -z''$. The direction cosines of the tangent line to L_0 at any point are $x' = X_u$, $y' = Y_u$, $z' = Z_u$, the arguments being (u,0). Let (l,m,n)

be the direction cosines of the principal normal to L_0, and let (a,b,c) represent the direction cosines of the binormal to L_0. If ρ is the radius of first curvature at any point on L_0, then we may define the positive direction on the binormal to be such that the relations

(10:2) $a = \rho(y'z'' - z'y'')$, $b = \rho(z'x'' - x'z'')$, $c = \rho(x'y'' - y'x'')$

hold at each point of L_0, where ρ is positive [6]. The positive direction of the principal normal is then fixed if the convention is made that the positive directions of the tangent line, principal normal and binormal at a point have the same mutual orientation as the positive directions of the x-, y-, z-axes, respectively. We then find the relations

(10:3) $\ell = \rho x'' = bz' - cy'$, $m = \rho y'' = cx' - az'$, $n = \rho z'' = ay' - bx'$.

The transversality condition stated in Theorem 6:2 then becomes $D(1 + pp_1 + qq_1)/f + (\Sigma \ell X_v)/\rho = 0$. By using equations (10:3) and (6:4), this may be written $(1 + pp_1 + qq_1)/f = (ap_1 + bq_1 - c)/\rho$. Let θ denote the angle between the normal line to the fixed surface and the normal line to the minimizing surface at any point on L_0, and let ψ denote the angle between the normal line to the fixed surface and the binormal to the curve L_0. The transversality condition is then

(10:4) $$\rho \cos \theta = \cos \psi \qquad (\theta \neq 0).$$

This relation must hold at each point on the boundary curve of a minimizing surface $z = z(x,y)$. A necessary and sufficient condition that the minimizing surface intersect the fixed surface orthogonally is that the boundary curve L_0 be a geodesic on the fixed surface. In this case the normal line to the fixed surface coincides with the principal normal to L_0, and in addition, the osculating plane of L_0 coincides with the tangent plane to the minimizing surface at each point of L_0. Therefore L_0 is an asymptotic curve on the minimizing surface.

The Euler equation for this problem is found by equation (6:9) to be $(1 + q^2)r - 2pqs + (1 + p^2)t = 0$, where p, q and r, s, t are the customary notations for the first and second partial derivatives of $z(x,y)$. The mean curvature of the minimizing surface is therefore zero, and the surface is a minimal surface.

The expression (7:10) for the second variation may be written in the form

$$I''(0) = \iint_{A_0} 2\omega \, dx\, dy + \int_0^{\chi} [2\Omega(\xi, \xi') + \frac{\zeta^2}{\sigma^2} C(u)]du,$$

where

$$2\omega(\zeta, \pi, \kappa) = [(1 + q^2)\pi^2 - 2pq\pi\kappa + (1 + p^2)\kappa^2]/f^3,$$
$$C(u) = AD_v + D(A_x X_v + A_y Y_v) - \Sigma X_{uu} X_{vv}.$$

In the expression for $C(u)$ the arguments of the derivatives of X, Y, Z are the set $(u,0)$. From equation (3:2) and the first equation in (7:6), we find the relations

$$D_v = (X_u Y_{vv} - Y_u X_{vv}) + (Y_v X_{uv} - X_v Y_{uv}),$$
$$A = -(pp_1 + qq_1 + 1)/f.$$

The quadratic form $2\Omega(\xi, \xi')$ reduces to a quadratic form in the three variables ξ_1', ξ_2', ξ_3' whose coefficients are the second derivatives of $g = \sqrt{x'^2 + y'^2 + z'^2}$ with respect to x', y', z', and has the explicit expression

$$2\Omega = (\xi_1'^2 + \xi_2'^2 + \xi_3'^2) - (\xi_1' X_u + \xi_2' Y_u + \xi_3' Z_u)^2.$$

By Lagrange's identity, the form 2Ω may be expressed as a sum of squares, and is positive.

The values of A_x and A_y may be found from equations (7:11). We first find the expressions

$$
\begin{aligned}
(f)_x &= (pr + qs)/f, & (f)_y &= (ps + qt)/f,\\
(f_p)_x &= [(1+q^2)r - pqs]/f^3, & (f_p)_y &= [(1+q^2)s - pqt]/f^3,\\
(f_q)_x &= [-pqr + (1+p^2)s]/f^3, & (f_q)_y &= [-pqs + (1+p^2)t]/f^3.
\end{aligned}
$$

We then obtain the values

$$
\begin{aligned}
A_x &= -(pr_1 + qs_1)/f + r[P(1+q^2) - Qpq]/f^3 + s[Q(1+p^2) - Ppq]/f^3,\\
A_y &= -(ps_1 + qt_1)/f + s[P(1+q^2) - Qpq]/f^3 + t[Q(1+p^2) - Pqp]/f^3,
\end{aligned}
$$

where $P = p - p_1$, $Q = q - q_1$. By means of the relations

$$
\begin{aligned}
P(1+q^2) - Qpq &= Pf^2 - p(pP + qQ) = -p_1f^2 + p(pp_1 + qq_1 + 1),\\
Q(1+p^2) - Ppq &= Qf^2 - q(pP + qQ) = -q_1f^2 + q(pp_1 + qq_1 + 1),
\end{aligned}
$$

the expressions for A_x and A_y may be written in the form

$$
\begin{aligned}
A_x &= -(pr_1 + qs_1 + rp_1 + sq_1)/f + (pp_1 + qq_1 + 1)(pr + qs)/f^3,\\
A_y &= -(ps_1 + qt_1 + sp_1 + tq_1)/f + (pp_1 + qq_1 + 1)(ps + qt)/f^3.
\end{aligned}
$$

The functions L, N, N occurring in the boundary value problem stated in Theorem 8:1 have the special values

$$
\begin{aligned}
L &= -H(u) = -C(u)/\sigma,\\
M &= D[P + q(pq_1 - qp_1)]/f^3,\\
N &= D[Q - p(pq_1 - qp_1)]/f^3.
\end{aligned}
$$

The Jacobi expression (7:3) has the form

$$
J(\zeta) = \frac{\partial}{\partial x}\left[\frac{pq\kappa - (1+q^2)\pi}{f^3}\right] + \frac{\partial}{\partial y}\left[\frac{pq\pi - (1+p^2)\kappa}{f^3}\right]
$$

For the classical problem of Plateau, $f = \sqrt{1 + p^2 + q^2}$, $g = 0$ and therefore $B = 0$. The transversality condition $A = 0$ is the orthogonality condition $pp_1 + qq_1 + 1 = 0$. The coefficients L, N, N in equation (8:4) have the values

$$\begin{aligned} L = D[&(pr_1 + qs_1 + rp_1 + sq_1)X_v \\ &+ (ps_1 + qt_1 + sp_1 + tq_1)Y_v]/\sigma f, \end{aligned}$$

$$M = - Dp_1/f, \qquad N = - Dq_1/f \ ,$$

where $\sigma = PX_v + QY_v$. If we make use of a normal coordinate system for which $\sigma X_v = P$, $\sigma Y_v = Q$, $\sigma^2 = P^2 + Q^2$, $D = 1$, a simpler form of the Jacobi condition for the problem of Plateau is obtained than that given by Simmons [12, p. 250].

At each element (x,y,z,p,q) of a minimizing surface S_0 the Weierstrass E-function [3, p. 27] has the value

$$E_1(x,y,z,p,q,\bar{p}, \bar{q}) = \frac{\sqrt{1 + p^2 + q^2}\cdot\sqrt{1 + \bar{p}^2 + \bar{q}^2} - (p\bar{p} + q\bar{q} + 1)}{\sqrt{1 + p^2 + q^2}},$$

where $(\bar{p},\bar{q},-1)$ represents an arbitrary direction at a point on S_0. The function E_1 is the same as that for the classical problem of Plateau, and is non-negative, vanishing only when $p = \bar{p}$, $q = \bar{q}$. This may readily be shown by considering the numerator of E_1 and applying Lagrange's identity. Along the boundary L_0 of a minimizing surface S_0 the Weierstrass E-function defined in equation (9:3) is found to have the value

$$E_2(X,X_u,X') = \frac{\sqrt{\Sigma X_u^2}\cdot\sqrt{\Sigma X'^2} - \Sigma X_u X'}{\sqrt{\Sigma X_u^2}} \ ,$$

where $X' = X_u + kX_v$, $Y' = Y_u + kY_v$, $Z' = Z_u + kZ_v$. By use of Lagrange's identity the numerator of this expression is readily shown to be non-negative, and to vanish only when $k = 0$. The direction (X', Y', Z') then coincides with the direction of the tangent line to L_0.

At every element (x, y, z, p, q) of a minimizing surface S_0 for the integral sum I the Legendre expression

$$f_{pp} \cos^2\alpha + 2f_{pq} \cos\alpha \sin\alpha + f_{qq} \sin^2\alpha$$

must be non-negative [3, p. 9] for every value of α. For the special problem of this section this expression may be put in the form $[1 + (p \sin\alpha - q \cos\alpha)^2]/f^3$. The strengthened form of the Legendre condition is therefore satisfied along S_0.

The Legendre expression given in equation (9:4) is found to have the value $\sum X_v^2 - (\sum X_u X_v)^2$. Let E, F, G denote the fundamental coefficients of the first order for the capstan surface (2:2), and let $H = \sqrt{EG - F^2}$. Then $E = 1$ along L_0 and the Legendre condition states that $H^2 \geqq 0$ along L_0. The inequality $H^2 > 0$ is always satisfied for a real surface without singular points.

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DIFFERENTIAL SYSTEMS WITH BOUNDARY CONDITIONS INVOLVING THE CHARACTERISTIC PARAMETER

BY

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DIFFERENTIAL SYSTEMS WITH BOUNDARY CONDITIONS INVOLVING THE CHARACTERISTIC PARAMETER

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1. Introduction. The present paper is concerned with the development of the theory of definitely self-adjoint systems of linear differential equations of the form

$$y_i' = \sum_{j=1}^{n} [A_{ij}(x) + \lambda B_{ij}(x)] y_j,$$

$$(1:1) \qquad s_i[y;\lambda] \equiv \sum_{j=1}^{n} [(M_{ij}^{0} + \lambda M_{ij}^{1}) y_j(a) + (N_{ij}^{0} + \lambda N_{ij}^{1}) y_j(b)] = 0$$

$$(a \leqq x \leqq b;\ i,j = 1,2,\ldots,n).$$

In 1926 Bliss [1]* gave definitions of self-adjointness and definite self-adjointness for a system of the form (1:1) in which the boundary conditions are independent of λ. For such definitely self-adjoint systems he showed that the characteristic numbers are real, infinite in number, and have indices equal to their multiplicities. He also obtained some very general expansion theorems. A weakened form of this definition is given in a rather recent paper [4] by the same author. For the boundary valu problems which are definitely self-adjoint according to this new definition it is true that the same expansion theorems hold, the characteristic numbers are real, and they have indices equal to their multiplicities. However, a boundary value problem may be definitely self-adjoint according to this modified definition, and not have an infinitude of characteristic values.

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[*] The numbers in brackets refer to the bibliography at the end of this paper.

In the following pages we shall extend the definitions and results of Bliss to a system of the form (1:1) which involves the parameter λ linearly in the boundary conditions.

In Section 2 some general properties of linear systems are given. Sections 3 and 4 are devoted to the discussion of adjoint and self-adjoint systems. A definition of definite self-adjointness is given in Section 5, and fundamental theorems concerning the characteristic values of such a system, analogous to those obtained by Bliss [4] for a system whose boundary conditions are independent of λ, are also proved in this section. Sections 6 and 7 deal with auxiliary theorems and lemmas which are basic to the development of the expansion theorems given in the following section. In Section 9 sufficient conditions for the existence of an infinity of characteristic values are given. Finally, in Section 10 it is shown that a certain boundary value problem of the calculus of variations of the type studied by Reid [6] is definitely self-adjoint according to the definition here given.

The methods used in the development of this theory are much the same as those used by Bliss in [1] and [4].

2. Preliminary remarks. In the first nine sections of this paper matrix notation will be used. Square matrices of n rows and columns are denoted by capital letters. For instance, M denotes the matrix whose element in the i-th row and j-th column is M_{ij} $(i,j = 1, \ldots, n)$. In particular, the zero and identity matrices are denoted by 0 and I, respectively. Similarly, the vector (v_i) $(i = 1, \ldots, n)$ is denoted by the lower case letter v. If $M \equiv \| M_{ij} \|$, $v \equiv (v_i)$, the vectors $(M_{ij}v_j)$ and $(v_j M_{ji})$ are denoted by Mv and vM, respectively, where the repetition of the subscript j indicates summation with respect to

this subscript over the range 1,2, ..., n. For the scalar product v_ju_j of the vectors $v \equiv (v_i)$, $u = (u_i)$ we shall write vu. The transpose of a matrix $M \equiv \| M_{ij} \|$ is denoted by $\tilde{M} \equiv \| M_{ji} \|$. If the elements of M are differentiable functions, M' signifies the corresponding matrix of derivatives; similarly, if the components of v are differentiable functions we write $v' \equiv (v_i')$. Finally, for brevity we write $M(\lambda)$ and $N(\lambda)$ for $M^0 + \lambda M^1$ and $N^0 + \lambda N^1$, respectively, when no confusion arises.

Concerning the boundary conditions

$$s[y; \lambda] \equiv M(\lambda)y(a) + N(\lambda)y(b) = 0$$

of the system (1:1) we shall make throughout the following hypothesis:

__For all values of__ λ __the__ $n \times 2n$ __matrix__ $\| M(\lambda); N(\lambda) \|$ __has rank__ n. __Moreover, there exist matrices__ $P(\lambda) = P^0 + \lambda P^1$, $Q(\lambda) = Q^0 + \lambda Q^1$, __together with matrices__ M^*, N^*, P^*, Q^* __independent of__ λ, __such that the__ $2n \times 2n$ __matrices__

$$(2:1) \qquad \left\| \begin{matrix} -P^* & -P(\lambda) \\ Q^* & Q(\lambda) \end{matrix} \right\| , \qquad \left\| \begin{matrix} M(\lambda) & N(\lambda) \\ M^* & N^* \end{matrix} \right\|$$

__are reciprocals for all values of__ λ.

This hypothesis is readily seen to be satisfied if there exists an $n \times n$ non-singular minor of $\| M(\lambda); N(\lambda) \|$ whose elements are independent of λ. It is also to be emphasized that the boundary conditions of our problem are unchanged if the matrices $M(\lambda)$, $N(\lambda)$ are replaced by $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$, respectively, where for each value of λ the matrix $\Gamma(\lambda)$ is non-singular; for our purpose, however, attention will be restricted to such matrices $\Gamma(\lambda)$ for which the resulting product matrices are linear in λ.

Since the matrices of (2:1) are reciprocals, it follows in particular that

$$M(\lambda)P(\lambda) = N(\lambda)Q(\lambda). \tag{2:2}$$

Consequently ([1], p. 562), the relations

$$t[z;\lambda] \equiv z(a)P(\lambda) + z(b)Q(\lambda) = 0 \tag{2:3}$$

are boundary conditions adjoint to the boundary conditions $s[y;\lambda] = 0$.

Since (2:2) is an identity in λ, it follows by equating equal powers of λ that

$$\begin{aligned} M^0P^0 &= N^0Q^0, \\ M^0P^1 + M^1P^0 &= N^0Q^1 + N^1Q^0, \\ M^1P^1 &= N^1Q^1. \end{aligned} \tag{2:4}$$

Similarly, one has also the following relations involving M^*, N^*, P^* and Q^*:

$$\begin{aligned} -P^*M^0 - P^0M^* &= I, & Q^*M^0 + Q^0M^* &= 0, \\ -P^*M^1 - P^1M^* &= 0, & Q^*M^1 + Q^1M^* &= 0, \\ -P^*N^0 - P^0N^* &= 0, & Q^*N^1 + Q^1N^* &= 0, \\ -P^*N^1 - P^1N^* &= 0, & Q^*N^0 + Q^0N^* &= I. \end{aligned} \tag{2:5}$$

If the matrices (2:1) are multiplied in the reverse order, then in addition to (2:2) the following further identities are obtained:

$$-M^*P(\lambda) + N^*Q(\lambda) = I, \tag{2:6}$$

$$-M(\lambda)P^* + N(\lambda)Q^* = I, \tag{2:7}$$

$$-M^*P^* + N^*Q^* = 0. \tag{2:8}$$

Now if we set

$$s^*[y] = M^*y(a) + N^*y(b),$$
$$t^*[z] = z(a)P^* + z(b)Q^*$$

it follows ([1], pp. 564-565) from equations (2:5) that we have the following identity in the arguments y(a), y(b), z(a), z(b) for all values of λ

$$t^*[z]s[y;\lambda] + t[z;\lambda]s^*[y] \equiv z(x)y(x)\Big|_a^b .$$

Consequently, if we write $s[y;\lambda] \equiv s^o[y] + \lambda s^1[y]$, $t[z;\lambda] \equiv t^o[z] + \lambda t^1[z]$, it follows that

$$(2:9) \qquad t^*[z]s^o[y] + t^o[z]s^*[y] \equiv z(x)y(x)\Big|_a^b ,$$

$$(2:10) \qquad t^*[z]s^1[y] + t^1[z]s^*[y] \equiv 0,$$

for arbitrary sets y(a), y(b), z(a), z(b). The above identities will be of fundamental importance in the development of the present work.

3. Adjoint systems of differential equations. The system (1:1) in matrix form may be written

$$(3:1) \qquad y' = [A(x) + \lambda B(x)]y, \qquad s[y;\lambda] = 0.$$

It will be assumed throughout that the elements of A(x), B(x) are real single-valued continuous functions on a finite interval $a \leqq x \leqq b$ and the elements of B(x) are not all identically zero on this interval; moreover, that the boundary conditions $s[y;\lambda] = 0$ satisfy the hypothesis of the preceding section. The system adjoint to (3:1) is by definition

$$(3:2) \qquad z' = -z[A(x) + \lambda B(x)], \qquad t[z;\lambda] = 0,$$

where the adjoint boundary conditions $t[z;\lambda] = 0$ are determined by the above equations (2:3).

The term characteristic solution will be used to denote a vector $y(x)$ whose components are not all identically zero on $a \leqq x \leqq b$, and which satisfies for a corresponding value λ the differential equations and two point boundary conditions of (3:1). If λ is a value for which there exist r linearly independent characteristic solutions, this value will be called a characteristic value of the system (3:1) of index r.

Let $Y(x, \lambda) \equiv \| Y_{ij}(x, \lambda) \|$ be a fundamental matrix solution of the differential equations of (3:1); that is, the columns of $Y(x, \lambda)$ form for each value of λ a set of n linearly independent solutions of these equations. Such a fundamental matrix solution could be determined by the initial value $Y(a, \lambda) = I$. It is readily seen that such a fundamental matrix solution has its elements expressible as permanently convergent power series in λ. The determinant $D(\lambda)$ of the matrix

$$(3:3) \qquad \| M(\lambda)Y(a,\lambda) + N(\lambda)Y(b,\lambda) \| \equiv \| s_i[Y_j] \|$$

is a characteristic determinant for the system (3:1) in the sense that its zeros are the characteristic values of that system. Since $M(\lambda)$ and $N(\lambda)$ are linear in λ, clearly $D(\lambda)$ is also a permanently convergent power series. It is readily seen that λ is a characteristic value of index r if and only if the matrix (3:3) has rank $n-r$. From general properties of differential systems (see Bliss [1], pp. 556-568) it also follows that if λ is a characteristic value of (3:1) of index r, then λ is also a characteristic value of the adjoint system (3:2) of index r.

Suppose now that $y(x)$ is a characteristic solution of the system (3:1) for a characteristic value λ_ν, and $z(x)$ is a characteristic solution of the adjoint system (3:2) for a characteristic value λ_2, $\lambda_2 \neq \lambda_1$. Then,

$$(3:1') \qquad y' = (A + \lambda_1 B)y, \qquad s^0[y] + \lambda_1 s^1[y] = 0,$$

$$(3:2') \qquad z' = -z(A + \lambda_2 B), \qquad t^0[z] + \lambda_2 t^1[z] = 0,$$

and by the use of the boundary conditions in (3:1') and (3:2'), together with the identity (2:9), it follows that

$$(\lambda_1 - \lambda_2) \int_a^b zBy \, dx = z(x)y(x)\Big|_a^b = -\lambda_1 t^*[z]s^1[y] - \lambda_2 t^1[z]s^*[y].$$

By using (2:10) the above expression can be reduced to the following equivalent conditions:

$$(3:4) \qquad t^*[z]s^1[y] + \int_a^b zBy \, dx = 0,$$

$$(3:4') \qquad -t^1[z]s^*[y] + \int_a^b zBy \, dx = 0,$$

$$(3:4'') \qquad \tfrac{1}{2}\Big[t^*[z]s^1[y] - t^1[z]s^*[y]\Big] + \int_a^b zBy \, dx = 0.$$

Thus we have the following result.

THEOREM 3:1. If y(x) is a solution of (3:1) for a value $\lambda = \lambda_1$, and z(x) is a solution of (3:2) for a value $\lambda = \lambda_2$, $\lambda_1 \neq \lambda_2$, then

$$(3:4) \qquad t^*[z]s^1[y] + \int_a^b zBy \, dx = 0.$$

4. Self-adjoint systems of differential equations. Follow- the definition of Bliss ([1], [4]) the boundary value problem (3:1) is said to be self-adjoint if the differential equations and the boundary conditions of its adjoint system are equivalent to its own for all values of λ under a transformation $z(x) = T(x)y(x)$,

where $T(x)$ is a non-singular matrix on $a \leqq x \leqq b$ with elements which are continuous and have continuous first derivatives on this interval. The proof of the following result is as in Bliss ([1], p. 569).

THEOREM 4:1. *The system* (3:1) *is self-adjoint if and only if there exists a transformation matrix* $T(x)$ *such that*

$$TA + \tilde{A}T + T' \equiv 0, \qquad TB + \tilde{B}T \equiv 0, \tag{4:1}$$

$$M(\lambda)T^{-1}(a)\tilde{M}(\lambda) = N(\lambda)T^{-1}(b)\tilde{N}(\lambda). \tag{4:2}$$

The above definition of self-adjointness, together with the results of Theorem 3:1, imply the following theorem.

THEOREM 4:2. *If the system* (3:1) *is self-adjoint with respect to a transformation matrix* $T(x)$, *and* $y(x)$ *and* $y^*(x)$ *are characteristic solutions corresponding to two distinct characteristic values* λ *and* λ^*, *then*

$$t^*[Ty^*]s^1[y] + \int_a^b y^*S(x)y\,dx = 0, \tag{4:3}$$

where $S(x) = \tilde{T}(x)B(x)$.

This follows from equation (3:4) and the fact that $z^* = Ty^*$ is a solution of the adjoint system for $\lambda = \lambda^*$.

5. *Definitely self-adjoint systems*. The definition of definite self-adjointness as used by Bliss will be extended to the system here considered as follows.

The boundary value problem (3:1) is *definitely self-adjoint* if the following hypotheses are satisfied.

(H_1) *It is self-adjoint with respect to the transformation matrix* $T(x)$.

(H_2) *The matrix* $S(x) = \tilde{T}(x)B(x)$ *is symmetric on* $a \leqq x \leqq b$.

(H_3) *The matrix*

$$(5:1)\qquad \left\|\begin{matrix} \tilde{T}(a)P^*M^1 & \tilde{T}(a)P^*N^1 \\ \tilde{T}(b)Q^*M^1 & \tilde{T}(b)Q^*N^1 \end{matrix}\right\|$$

belonging to the quadratic form $t^*[Ty]s^1[y] \equiv \mathfrak{Q}[y(a),y(b)] \equiv Q[y]$ *is symmetric*. The corresponding bilinear form $t^*[Tv]s^1[u]$ in the arguments $v(a)$, $v(b)$, $u(a)$, $u(b)$ we shall denote by $\mathfrak{Q}[v(a),v(b);u(a),u(b)]$, or $\mathfrak{Q}[u;v]$ for brevity. Clearly $\mathfrak{Q}[v;v] = Q[v]$.

(H_4) *The quadratic expression*

$$K[y] \equiv \mathfrak{Q}[y(a),y(b)] + \int_a^b y(x)S(x)y(x)\,dx$$

is non-negative on $a \leqq x \leqq b$ *for arbitrary continuous* $y(x)$. The corresponding bilinear expression

$$\mathfrak{Q}[u(a),u(b);v(a),v(b)] + \int_a^b uSv\,dx$$

will be denoted by $K[u;v]$. Clearly $K[u;u] = K[u]$.

(H_5) *For an arbitrary value* λ, *the only solution of the system* (3:1) *satisfying* $K[y] = 0$ *on* $a \leqq x \leqq b$ *is* $y(x) \equiv 0$.

As immediate consequences of the above definition we have some important results which we state here as lemmas.

LEMMA 5:1. *Hypotheses* (H_2) *and* (H_4) *imply that the quadratic form* $yS(x)y$ *is positive semi-definite on* $a \leqq x \leqq b$.

For suppose that the conclusion of the lemma is not true. Then there exists some vector u and a value x_0, $a < x_0 < b$, such that $uS(x_0)u < 0$. The continuity of the elements of $S(x)$ imply that there exists a $d > 0$ such that $uS(x)u < 0$ for $|x - x_0| < d$. If we now define $y(x) = u\,\xi(x)$, where $\xi(x)$ is a continuous function such that $\xi(x) \neq 0$ on $x_0 - d < x < x_0 + d$ and $\xi(x) \equiv 0$ except on this interval, we see that $yS(x)y = uS(x)u\,\xi^2(x) < 0$ on

$x_0 - d < x < x_0 + d$, $yS(x)y \equiv 0$ elsewhere. Then

$$K[y] = \int_{x_0 - d}^{x_0 + d} uS(x)u\,\xi^2(x)\,dx < 0$$

and this is a contradiction to hypothesis (H_4).

LEMMA 5:2. Hypotheses (H_3) and (H_4) imply that the quadratic form $Q[y(a), y(b)]$ is positive semi-definite.

For suppose that there exists a vector $u(x)$ such that $Q[u] \equiv Q[u(a), u(b)] < 0$. Define a function $\xi(x)$ in the following manner: $\xi(x) \equiv 0$ on $a + d \leqq x \leqq b - d$, $\xi(a) = \xi(b) = 1$ and $\xi(x)$ linear on $a \leqq x \leqq a + d$ and $b - d \leqq x \leqq b$. If we set $y(x) = u(x)\,\xi(x)$ we obtain

$$K[y] = Q[u] + \int_a^{a+d} uSu\,\xi^2\,dx + \int_{b-d}^{b} uSu\,\xi^2\,dx.$$

Since the two integrals tend to zero as d approaches zero and $Q[u] < 0$, for d sufficiently small $y(x)$ is such that $K[y] < 0$, which is a contradiction to (H_4). Hence Q is a positive semi-definite form.

LEMMA 5:3. If for a continuous $y(x)$ we have $K[y] = 0$, then hypotheses (H_2), (H_3) and (H_4) imply $B(x)y(x) \equiv 0$ on $a \leqq x \leqq b$, $s^1[y] = 0$.

Lemmas 5:1 and 5:2 and the condition that $K[y] = 0$ imply that $ySy \equiv 0$ on $a \leqq x \leqq b$. This says that $Sy \equiv 0$, and therefore, since the matrix $T(x)$ is non-singular, it follows that $By \equiv 0$ on $a \leqq x \leqq b$. The condition $Q[y(a), y(b)] = 0$ is then a consequence of Lemma 5:2 and the relation $K[y] = 0$. If we now take the partial derivatives of the above quadratic form with respect to the arguments $y(a)$, $y(b)$, and remember that its matrix of coefficients (5:1) is symmetric, we have that $P^* s^1[y] = 0$, $Q^* s^1[y] = 0$. These two relations and the fact that the rank of $\left\| \begin{matrix} P^* \\ Q^* \end{matrix} \right\|$ is n, as implied by equation (2:5), justify the last result of the lemma.

It is therefore seen that this lemma enables us to state that whenever hypotheses (H_2), (H_3) and (H_4) are satisfied hypothesis (H_5) is equivalent to:

(H_5') The only solution of

$$y' = A(x)y, \qquad s^0[y] = 0$$

satisfying $ySy \equiv 0$ on $a \leqq x \leqq b$, $s^1[y] = 0$ is $y \equiv 0$.

In view of Lemmas 5:1 and 5:2 it follows that $K[f] \geqq 0$ for arbitrary vectors $f(x)$ whose components are bounded and integrable on $a \leqq x \leqq b$. In most of the following discussion the arguments of the quadratic expression K will be vectors whose components are continuous on this interval. In Sections 7 and 8, however, we will have occasion to use arguments whose elements are discontinuous. In view of the non-negativeness of K we have for arbitrary vectors $f(x)$, $g(x)$ whose components are bounded and integrable on $a \leqq x \leqq b$ the following inequality

$$\{K[f;g]\}^2 \leqq K[f] \cdot K[g]. \tag{5:2}$$

If we consider again equation (4:2) we see that its transpose, together with equation (2:2), justify relations of the form

$$\begin{aligned} \tilde{T}^{-1}(a)\tilde{M}(\lambda) &= P(\lambda)C(\lambda) \\ \tilde{T}^{-1}(b)\tilde{N}(\lambda) &= Q(\lambda)C(\lambda) \end{aligned} \tag{5:3}$$

where $C(\lambda)$ is a non-singular matrix for all values of λ. The above equations (5:3) are written to simplify the statement of the following important lemma.

LEMMA 5:4. Under the hypothesis (H_3) the matrix $C(\lambda)$ defined by equations (5:3) is independent of λ.

The proof of this lemma is as follows. Multiply equations (5:3) on the left by $-M^*$ and N^*, respectively, and add. Use of equation (2:6) then yields

$$C(\lambda) = -M^*\tilde{T}^{-1}(a)\tilde{M}(\lambda) + N^*\tilde{T}^{-1}(b)\tilde{N}(\lambda). \tag{5:4}$$

Consequently, $C(\lambda)$ is linear in λ and we may write $C(\lambda) = C^0 + \lambda C^1$.

By hypothesis (H_3) the matrix of coefficients (5:1) is symmetric, and therefore

$$\tilde{T}(a)P^*M^1 = \tilde{M}^1\tilde{P}^*T(a), \tag{5:5}$$

$$\tilde{T}(b)Q^*N^1 = \tilde{N}^1\tilde{Q}^*T(b), \tag{5:6}$$

$$\tilde{T}(a)P^*N^1 = \tilde{M}^1\tilde{Q}^*T(b). \tag{5:7}$$

If we multiply equations (5:7) and (5:6) on the left by $-M^*\tilde{T}^{-1}(a)$ and $N^*\tilde{T}^{-1}(b)$, respectively, and add, it follows by equations (5:4), (2:8) that

$$Q^*\tilde{C}^1 = 0. \tag{5:8}$$

Multiply now equations (5:5) and (5:7) on the right by $-T^{-1}(a)\tilde{M}^*$ and $T^{-1}(b)\tilde{N}^*$, respectively, and add. An application of equations (5:4) and (2:8) shows that

$$P^*\tilde{C}^1 = 0. \tag{5:9}$$

Finally, if we multiply (5:9) and (5:8) on the left by $-M^0$ and N^0, respectively, and add, it follows in view of relation (2:7) that $C^1 = 0$.

The following theorems are concerned with certain important general properties of definitely self-adjoint systems.

THEOREM 5:1. *The characteristic values of a definitely self-adjoint boundary value problem* (3:1) *are real and the corresponding characteristic solutions may be chosen real.*

For suppose $y = y^1 + \sqrt{-1}\, y^2$ were a characteristic solution for $\lambda = \lambda_1 + \sqrt{-1}\, \lambda_2$, $\lambda_2 \neq 0$. Then the complex conjugate $\bar{y} = y^1 - \sqrt{-1}\, y^2$ would be a solution for $\lambda = \bar{\lambda}$ and Theorem 4:2 would imply that $K[y; \bar{y}] = 0$. By a property of quadratic forms the above expression reduces to $K[y^1] + K[y^2] = 0$; that is, $K[y^1] = K[y^2] = 0$. By the use of Lemma 5:3 it would then follow that y^1 and y^2 were individually solutions of (3:1) for $\lambda = 0$, and since $K[y^1] = K[y^2] = 0$ we would have in view of (H_5) that $y^1 \equiv 0$, $y^2 \equiv 0$. This, however, would imply $y \equiv y^1 + \sqrt{-1}\, y^2 \equiv 0$, which contradicts the assumption that y was a characteristic solution. Therefore the λ's are real and the characteristic solutions may be chosen real.

THEOREM 5:2. *For a definitely self-adjoint boundary value problem* (3:1) *the index of a root of* $D(\lambda)$ *is equal to its multiplicity.*

If $\lambda = \lambda_0$ is a characteristic value of index r, for this value of λ the matrix (3:3) has rank $n - r$. By a proper choice of linear combinations of solutions the columns of this matrix can be so arranged as to make $s[Y_p; \lambda_0] = 0$ $(p = 1, 2, \ldots, r)$. Since $D(\lambda)$ is the determinant of this matrix, at $\lambda = \lambda_0$ the derivatives of $D(\lambda)$ of order less than r will clearly vanish. The r-th derivative will have the following form

$$(5:10) \qquad D^{(r)}(\lambda_0) = r! \left| s[Y_{h\lambda}; \lambda_0] + s^1[Y_h] \qquad s[Y_m; \lambda_0] \right|$$

$$(h = 1, \ldots, r;\ m = r + 1, \ldots, n).$$

Suppose now that the above determinant vanishes. Then there exist constants c_h, c_m $(h = 1,\ldots,r;\ m = r+1,\ldots,n)$ not all zero and such that the equation

$$(5{:}11) \qquad \sum_{h=1}^{r} s[Y_{h\lambda}; \lambda_0]c_h + \sum_{m=r+1}^{n} s[Y_m; \lambda_0]c_m = -\sum_{h=1}^{r} s^1[Y_h]c_h$$

is satisfied. If we set

$$y^1(x) = \sum_{h=1}^{r} Y_h(x, \lambda_0)c_h, \qquad y^2(x) = \sum_{m=r+1}^{n} Y_m(x, \lambda_0)c_m$$

$$y^3(x) = \sum_{h=1}^{r} Y_{h\lambda}(x, \lambda_0)c_h, \qquad y(x) = y^3(x) + y^2(x),$$

it follows that

$$(5{:}12) \qquad s[y; \lambda_0] = -s^1[y^1].$$

Now the constants c_h $(h = 1,\ldots,r)$ can not all be zero since the last $n - r$ columns of the determinant (5:10) are linearly independent. It follows in particular, therefore, that $y^1 \not\equiv 0$. Since $y(x,\lambda) = \sum_{h=1}^{r} Y_h(x,\lambda)c_h$ satisfies (3:1) for all values of λ, it follows on differentiating with respect to λ and evaluating at $\lambda = \lambda_0$ that

$$y^{3\prime} = (A + \lambda_0 B)y^3 + By^1.$$

As y^2 is a solution of (3:1) for $\lambda = \lambda_0$ we have, therefore, that

$$y' = (A + \lambda_0 B)y + By^1, \qquad s[y; \lambda_0] = -s^1[y^1].$$

Finally, as $z^1 = Ty^1$ satisfies equation (3:2) for $\lambda = \lambda_0$, it follows ([1], pp. 566-568; [6], pp. 776-777) that $K[y^1] = 0$. Hypothesis (H_5) then requires y^1 to vanish identically, which is contrary to the above established conclusion that $y^1 \not\equiv 0$ on $a \leqq x \leqq b$. We have thus been led to a contradiction on the assumption that the derivative $D^{(r)}(\lambda_0)$ is equal to zero. Hence this derivative is different from zero, and the multiplicity of λ_0 is equal to its index.

THEOREM 5:3. If for a given continuous $f(x)$ the condition

$$K[y;f] = 0 \tag{5:13}$$

is satisfied by every characteristic solution $y(x)$ of the definitely self-adjoint problem (3:1), this condition also holds for every $y(x)$ which satisfies with a continuous $g(x)$ the system

$$y' = Ay + Bg, \qquad s^0[y] = -s^1[g]. \tag{5:14}$$

The condition (5:13) implies ([1], pp. 566-568; [6], pp. 776-777) that the non-homogeneous system

$$y' = (A + \lambda B)y + Bf, \qquad s[y;\lambda] = -s^1[f] \tag{5:15}$$

has solutions for every value of λ. If $D(\lambda) \neq 0$ there is a unique solution which is of the form

$$y(x,\lambda) = Y_0(x,\lambda) + \sum_{j=1}^{n} Y_j(x,\lambda)c_j,$$

where $y = Y_0(x,\lambda)$ is a particular solution of the differential equations of the system (5:15). It is readily seen that the components of $y(x,\lambda)$ are expressible in the form

$$y_i(x,\lambda) = \frac{1}{D(\lambda)} \begin{vmatrix} Y_{i0}(x,\lambda) & Y_{i1}(x,\lambda) & \cdots & Y_{in}(x,\lambda) \\ s_1[Y_0;\lambda] + s_1^1[f] & s_1[Y_1;\lambda] & \cdots & s_1[Y_n;\lambda] \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ s_n[Y_0;\lambda] + s_n^1[f] & s_n[Y_1;\lambda] & \cdots & s_n[Y_n;\lambda] \end{vmatrix} \tag{5:16}$$

In a neighborhood of a zero $\lambda = \lambda_0$ of $D(\lambda)$ the solution $y(x,\lambda)$

is well defined and analytic in λ. For in the determinant (5:16) we can add multiples of the last n columns to the first column so as to make Y_0 satisfy $s[Y_0; \lambda] = -s^1[f]$ at the root $\lambda = \lambda_0$. Furthermore, if the multiplicity of λ_0 is equal to r, the r linearly independent solutions $z(x) = z(x, \lambda_0)$ of the adjoint system (3:2) for $\lambda = \lambda_0$ afford r linearly independent sets $t^*[z]$ satisfying $t^*[z]s[Y_j; \lambda_0] = 0$ $(j = 1,2,\ldots,n)$. Then r of the last n rows of (5:16) can be replaced by suitable linear combinations which vanish at $\lambda = \lambda_0$, and consequently the determinant in (5:16) has the same factor $(\lambda - \lambda_0)^r$ as does $D(\lambda)$. Hence the vector $y(x,\lambda)$ is analytic and well defined near a zero $\lambda = \lambda_0$ of $D(\lambda)$. Consequently $y(x,\lambda)$ is representable as a permanently convergent power series in λ of the form

$$(5:17)\quad y(x,\lambda) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \ldots + \lambda^n u_n(x) + \cdots .$$

If we substitute (5:17) in (5:15) and equate coefficients of like powers of λ, it follows that

$$(5:18)\quad u_q' = Au_q + Bu_{q-1}, \qquad s^0[u_q] + s^1[u_{q-1}] = 0 \quad (q = 0,1, \ldots),$$

where it is to be understood that $u_{-1} = f$. If the functions v_p are defined by $v_p = Tu_p$ $(p = -1,0,1,2,\ldots)$ it may be shown that

$$(5:19)\quad v_p' = -v_pA - v_{p-1}B, \quad t^0[v_p] + t^1[v_{p-1}] = 0 \qquad (p = 0,1, \ldots).$$

The differential equations (5:19) follow by direct substitution and use of conditions (4:1) satisfied by the transformation matrix T. The boundary conditions are a consequence of relation (4:2); in particular, use is made of the result of Lemma 5:3 concerning the independence of λ of the matrix C defined by (5:3). Equations (5:18), (5:19), (2:9) and (2:10) then lead to an expression of the form

$$\int_a^b v_p B u_{q-1}\,dx - \int_a^b v_{p-1} B u_q\,dx = u_q v_p\Big|_a^b = -t^*[v_p]\,s^1[u_{q-1}] + t^*[v_{p-1}]\,s^1[u_q]$$

that is,

$$K[u_{q-1};u_p] = K[u_q;u_{p-1}] \qquad (p,q = 0,1,2,\ldots). \tag{5:20}$$

Now as a special case of (5:2) we have

$$\left\{K[u_{q+1};u_{q-1}]\right\}^2 \leqq K[u_{q+1}]\cdot K[u_{q-1}]. \tag{5:21}$$

If we define

$$W_q = K[u_0;u_q] \qquad (q = 0,1,2,\ldots)$$

equations (5:20) and (5:21) show that

$$W_{q+p} = K[u_q;u_p], \qquad W_{2q-2}W_{2q+2} \geqq W_{2q}^2 . \tag{5:22}$$

Now the series (5:17) converges absolutely and uniformly for $a \leqq x \leqq b$, and λ in a bounded region of the complex λ-plane. Hence we can compute $\int_a^b u_0 Sy\,dx$ by term by term integration. In particular, the expression $K[u_0;y]$ is given by the permanently convergent power series

$$W_0 + \lambda W_1 + \lambda^2 W_2 + \ldots + \lambda^n W_n + \ldots .$$

It then follows that the series involving only even powers of λ,

$$W_0 + \lambda^2 W_2 + \lambda^4 W_4 + \ldots , \tag{5:23}$$

is also permanently convergent. Now some coefficient in (5:23) must vanish. For otherwise, since $W_{2q} \geqq 0$ and $W_{2q-2}W_{2q+2} \geqq W_{2q}^2$, it would follow that

$$\frac{W_0}{W_2} \geqq \frac{W_2}{W_4} \geqq \frac{W_4}{W_6} \geqq \ldots ,$$

and for $\lambda = (W_0/W_2)^{1/2}$ the terms of (5:23) are not less than the corresponding terms of the divergent series

$$W_0 + W_0 + W_0 + \cdots .$$

Hence some coefficient in (5:23) is zero, and by successive applications of the inequality of (5:22) it follows that $K[u_0] = W_0 = 0$. Moreover, equation (5:18) for $q = 0$ gives

$$(5{:}24) \qquad u_0' = Au_0 + Bf, \qquad s^0[u_0] = -s^1[f].$$

Now suppose that $y = y(x)$ satisfies (5:14) with a continuous $g(x)$. Then $z = Ty$ satisfies the system

$$(5{:}25) \qquad z' = -zA - g\tilde{T}B, \qquad t^0[z] = -t^1[g\tilde{T}].$$

It then follows by the use of (2:9), (2:10) and the result of Lemma 5:3 that

$$(5{:}26) \qquad K[y;f] = K[u_0;g].$$

Now the right member of (5:26) is zero because of the condition $K[u_0] = 0$ and the inequality $\{K[u_0;g]\}^2 \leqq K[u_0]\cdot K[g] = 0$ implied by (5:2). Consequently, if f satisfies (5:13) with every characteristic solution y of the definitely self-adjoint system (3:1) we also have $K[y;f] = 0$ for every $y(x)$ satisfying the system (5:14) with a continuous $g(x)$.

COROLLARY 1. *If* $f(x)$ *satisfies condition* (5:13) *with every characteristic solution of the definitely self-adjoint system* (3:1), *and in addition* $f(x)$ *satisfies with a continuous* $g(x)$ *the system*

$$(5{:}27) \qquad f' = Af + Bg, \qquad s^0[f] = -s^1[g],$$

then $K[f] = 0$, $Bf \equiv 0$ *on* $a \leqq x \leqq b$, *and* $s^1[f] = 0$.

The first conclusion of the corollary follows immediately from the fact that $y = f(x)$ satisfies equation (5:27), which is of the same form as (5:14); the conditions $Bf \equiv 0$, $s^1[f] = 0$ are consequences of Lemma 5:3.

Since for a characteristic solution y we have $K[y] \neq 0$ by (H_5), the different characteristic solutions corresponding to each characteristic value λ may be normed and orthogonalized. Consequently, we may denote by $\{\lambda_q\}$, $\{y_q(x)\}$ the totality of characteristic values and corresponding characteristic solutions, the latter orthonormal in the sense that

$$(5:28) \qquad \begin{aligned} K[y_q; y_p] &= 1, \text{ if } q = p, \\ &= 0, \text{ if } q \neq p. \end{aligned}$$

6. *The Green's matrix.* Suppose that the homogeneous system

$$(6:1) \qquad y' = Ay, \qquad s^o[y] \equiv M^o y(a) + N^o y(b) = 0$$

is incompatible, that is, that $\lambda = 0$ is not a characteristic value of (3:1). Since for a definitely self-adjoint system (3:1) not all values of λ are characteristic values in view of Theorem 5:2, this condition may always be obtained by a linear change of the characteristic parameter. Then there exists a unique Green's matrix $G(x,t) = \| G_{ij}(x,t) \|$ for this system. For a detailed discussion of the Green's matrix and its properties the reader is referred to Bliss [1], Section 5. Its explicit form is

$$G(x,t) = \frac{1}{2} Y(x) \left[\frac{|x-t|}{x-t} I + D^{-1} \Delta \right] Y^{-1}(t)$$

where $D = M^o Y(a) + N^o Y(b)$, $\Delta = M^o Y(a) - N^o Y(b) = 0$, and $Y(x)$ is a fundamental matrix solution of the differential equations of the system (6:1). For $a < t < b$ the matrix $G(x,t)$ has the following fundamental properties as a function of x:

$$G(t+0,t) - G(t-0,t) = I, \tag{6:2}$$

$$\frac{\partial}{\partial x} G(x,t) = A(x)G(x,t) \quad \text{on } a < x < t \text{ and } t < x < b, \tag{6:3}$$

$$M^oG(a,t) + N^oG(b,t) = 0. \tag{6:4}$$

Similarly, for $a < x < b$ this matrix has the following properties as a function of t:

$$G(x,x-0) - G(x,x+0) = I, \tag{6:5}$$

$$\frac{\partial}{\partial t} G(x,t) = -G(x,t)A(t) \quad \text{on } a < t < x \text{ and } x < t < b, \tag{6:6}$$

$$G(x,a)P^o + G(x,b)Q^o = 0. \tag{6:7}$$

Use will now be made of the Green's matrix to establish equivalence relations between our differential system and a certain type of integral equation.

THEOREM 6:1. *For an arbitrary continuous* $g(x)$ *the non-homogeneous system*

$$y' = Ay + Bg, \qquad s^o[y] = -s^1[g] \tag{6:8}$$

is equivalent to the integral equation

$$y(x) = t^*[G(x)]s^1[g] + \int_a^b G(x,t)B(t)g(t)dt, \tag{6:9}$$

where $t^*[G(x)]$ *represents* $G(x,a)P^* + G(x,b)Q^*$.

Equations (6:8) and (6:6) show that

$$\frac{\partial}{\partial t} [G(x,t)y(t)] = G(x,t)B(t)g(t)$$

on $a < t < x$ and $x < t < b$. Therefore,

$$G(x,x-0)y(x) - G(x,a)y(a) = \int_a^x G(x,t)B(t)g(t)\,dt,$$

$$G(x,b)y(b) - G(x,x+0)y(x) = \int_x^b G(x,t)B(t)g(t)\,dt.$$

Addition of the two previous equations and use of the equations (2:9), (2:10), (6:5) and (6:7) leads to the relation (6:9).

Conversely, suppose $y(x)$ is defined by (6:9); that is,

$$(6{:}10) \quad y(x) = t^*[G(x)]\,s^1[g] + \int_a^x G(x,t)B(t)g(t)\,dt + \int_x^b G(x,t)B(t)g(t)$$

Using (6:3) and (6:5) it readily follows that $y' = Ay + Bg$. Moreover, by the explicit form of $G(x,t)$ it is seen that

$$\lim_{x\to a} G(x,a) = \frac{1}{2}\,Y(a)[I + D^{-1}\Delta\,]Y^{-1}(a),$$

$$\lim_{x\to b} G(x,a) = \frac{1}{2}\,Y(b)[I + D^{-1}\Delta\,]Y^{-1}(a),$$

$$\lim_{x\to a} G(x,b) = \frac{1}{2}\,Y(a)[-I + D^{-1}\Delta\,]Y^{-1}(b),$$

$$\lim_{x\to b} G(x,b) = \frac{1}{2}\,Y(b)[-I + D^{-1}\Delta\,]Y^{-1}(b).$$

Direct substitution, together with the equation (2:7), yields $s^o[y] = -s^1[g]$. Thus our theorem is proved.

COROLLARY 1. _If $\lambda = 0$ is not a characteristic value of the boundary value problem_ (3:1), _this system is equivalent to the integral equation_

$$(6{:}11) \qquad y(x) = \lambda t^*[G(x)]\,s^1[y] + \lambda\int_a^b G(x,t)B(t)y(t)\,dt.$$

This integral equation is not of the usual kind, since it involves outside the integral sign the end values of $y(x)$ at $x = a$ and $x = b$.

7. *Auxiliary lemmas.* The definition of definite self-adjointness as given in Section 5 of this paper does not furnish sufficient conditions for the existence of infinitely many characteristic values. Therefore, it will be understood in the following theorems that the range of the summation indices will be finite or infinite according as the problem has a finite or an infinite number of characteristic values. In the following it is assumed that the $y_q(x)$ $(q = 1, 2, \ldots)$ are the totality of characteristic solutions of a definitely self-adjoint boundary value problem (3:1). We shall also suppose that these characteristic solutions are orthonormal in the sense of (5:28).

If now for a continuous $f(x)$ generalized Fourier coefficients $c_q = c_q[f]$ are defined as $c_q = K[f; y_q]$, for an arbitrary integer m the orthonormal character of the y_q implies

$$K[f - \sum_{q=1}^{m} y_q c_q] = K[f] - \sum_{q=1}^{m} c_q^2.$$

The following lemma is, therefore, an immediate consequence of hypothesis (H_4).

LEMMA 7:1. *For a continuous* $f(x)$ *the series* $\sum_q c_q^2$ *converges.*

LEMMA 7:2. *Suppose that the components of* $h = h(x,t)$ *are bounded for all values of* x *and* t *on the interval* ab, *and integrable in* t *on that interval for every fixed* x; *moreover, that the components of* g *are continuous on this interval. If* $d_q = K[y_q; g]$, $\chi_q(x) = K[y_q; h]$, *where* $\chi_q(x)$ *is computed considering* h *as a function of* t, *then the series*

$$\sum_q \chi_q(x) d_q \tag{7:1}$$

converges absolutely and uniformly on the interval $a \leqq x \leqq b$.

For arbitrary positive integral values p and m, let

$$\sum_{q=p}^{p+m} |\chi_q(x)d_q| = \sum \chi_\rho (x)d_\rho - \sum \chi_\sigma (x)d_\sigma ,$$

where ρ takes on those integral values for which $\chi_\rho\ d_\rho$ is positive and σ those for which $\chi_\sigma\ d_\sigma$ is negative. Then

$$\begin{aligned} \sum_\rho \chi_\rho (x)d_\rho &= \sum_\rho K[h; y_\rho]d_\rho \\ &= K[h; \sum_\rho y_\rho\ d_\rho] \\ &\leqq \left\{K[h] \sum_\rho d_\rho^2\right\}^{1/2} \end{aligned}$$

in view of (5:2) and the orthonormal character of the y_q. Consequently,

$$\sum_{\rho \geqq p} \chi_\rho (x)d_\rho \leqq R^{1/2}\Big(\sum_{\rho \geqq p} d_\rho^2\Big)^{1/2}$$

where R is the maximum value on $a \leqq x \leqq b$ of K[h] computed considering h as a function of t. Similarly,

$$-\sum_{\sigma \geqq p} \chi_\sigma (x)d_\sigma \leqq R^{1/2}\Big(\sum_{\sigma \geqq p} d_\sigma^2\Big)^{1/2}.$$

Therefore,

$$\sum_{q=p}^{p+m} |\chi_q(x)d_q| \leqq 2R^{1/2}\Big(\sum_{q=p}^{p+m} d_q^2\Big)^{1/2},$$

and since $\sum_q d_q^2$ converges by Lemma 7:1 it follows that $\sum_q \chi_q(x)d_q$ converges absolutely and uniformly on $a \leqq x \leqq b$.

8. Expansion theorems. The orthonormal character of the characteristic solutions of the definitely self adjoint boundary value problem (3:1), together with the theorems proved in the preceding sections, justify the following important expansion theorems.

THEOREM 8:1. If $f(x)$ satisfies the system

$$f' = Af + Bg, \qquad s^0[f] = -s^1[g] \tag{8:1}$$

with a continuous $g(x)$, then

$$K[y_q;g] = \lambda_q K[y_q;f] \qquad (q = 1,2, \dots).$$

This result is obtained immediately from (8:1), (3:2), (2:9), (2:10) and the fact that $z_q = Ty_q$ is a solution of the adjoint system for $\lambda = \lambda_q$.

THEOREM 8:2. If $f(x)$ satisfies the system (8:1) with a continuous $g(x)$, and $c_q = K[y_q;f]$, the series

$$\phi(x) = \sum_q y_q(x)c_q \tag{8:2}$$

converges absolutely and uniformly, and

$$K[y_p;f - \phi] = 0 \qquad (p = 1,2, \dots), \tag{8:3}$$

$$B(f - \phi) \equiv 0 \text{ on } a \leqq x \leqq b, \quad s^1[f - \phi] = 0. \tag{8:4}$$

If $d_q = K[y_q;g]$, it follows from Theorem 8:1 that $d_q = \lambda_q c_q$, and hence

$$\phi(x) = \sum_q \frac{y_q}{\lambda_q} d_q.$$

In view of the integral equation (6:11) we then have that $\phi_i(x)$, the i-th component of $\phi(x)$, is given by a series of the form (7:1), where the corresponding vector $h(x,t)$ is the i-th row of the matrix $G(x,t)\tilde{T}^{-1}(t)$. It then follows by Lemma 7:2 that the series representing $\phi_i(x)$ converges absolutely and uniformly on $a \leqq x \leqq b$.

We may then calculate $K[y_p; f - \phi]$ by term-wise evaluation. It follows that for each p,

$$K[y_p; f - \emptyset] = K[y_p; f] - \sum_q K[y_p; y_q]c_q = K[y_p; f] - K[y_p; f] = 0.$$

This relation, together with the definition of $\emptyset(x)$, is seen to imply that

$$K[\emptyset; f - \emptyset] = 0. \tag{8:5}$$

An application of Theorem 5:3 now justifies the relation

$$K[f; f - \emptyset] = 0, \tag{8:6}$$

and by subtracting equation (8:5) from (8:6) we find that $K[f - \emptyset] = 0$. By Lemma 5:3 it now follows that $B(f - \emptyset) \equiv 0$ on $a \leqq x \leqq b$ and $s^1[f - \emptyset] = 0$.

COROLLARY 1. *If the matrix* $B(x)$ *is non-singular on* $a \leqq x \leqq b$, *then for every continuous* $f(x)$ *satisfying* (8:1) *with a continuous* $g(x)$ *the series* (8:2) *converges absolutely and uniformly on* $a \leqq x \leqq b$ *and represents* $f(x)$.

THEOREM 8:3. *If* $f(x)$ *satisfies with a continuous* $g(x)$ *the system* (8:1), *and* $g(x)$ *in turn satisfies the system*

$$g' = Ag + Bk, \qquad s^0[g] = -s^1[k] \tag{8:7}$$

with a continuous $k(x)$, *then the series* (8:2) *converges absolutely and uniformly on* $a \leqq x \leqq b$ *and represents* $f(x)$.

Theorem 8:1 justifies the relation

$$\sum_q \lambda_q y_q K[y_q; f] = \sum_q y_q K[y_q; g]. \tag{8:8}$$

Since $g(x)$ satisfies equation (8:7), which is similar to (8:1), an application of the previous theorems shows that the left member of equation (8:8) converges absolutely and uniformly on $a \leqq x \leqq b$. Now set $\theta(x) = \sum_q y_q(x)K[y_q; g]$. If we differentiate (8:2) with respect to x and substitute the values of y' from (3:1), it follows that

$$(8:9) \qquad \phi' = A\phi + B\theta, \qquad s^0[\phi] = -s^1[\theta].$$

From systems (8:1) and (8:9) we therefore obtain

$$(8:10) \qquad f' - \phi' = A(f-\phi) + B(g-\theta), \qquad s^0[f-\phi] = -s^1[g-\theta].$$

Since g satisfies the system (8:7) it is a consequence of Theorem 8:2, applied to g in place of f, that $B(g-\theta) \equiv 0$, $s^1[g-\theta] = 0$. Therefore,

$$(8:10') \qquad f' - \phi' = A(f-\phi), \qquad s^0[f-\phi] = 0.$$

This latter relation, in view of relations (8:4) and (H_5), or its alternative form (H_5'), implies that $f - \phi \equiv 0$ on $a \leqq x \leqq b$.

9. Sufficient conditions for the existence of infinitely many characteristic values. As pointed out in the preceding section, the definition of definite self-adjointness which has been formulated in this paper does not imply the existence of infinitely many characteristic values for a system (3:1). Here we shall give a sufficiency theorem by introducing the further hypothesis:

(H_6) *For arbitrary continuous* $g(x)$ *the only solution of the system*

$$y' = Ay + Bg, \qquad s^0[y] = -s^1[g]$$

for which $K[y] = 0$ is $y \equiv 0$ on $a \leqq x \leqq b$.

It is easily seen that hypothesis (H_6) implies (H_5). Hypothesis (H_6) corresponds to a condition imposed by Bliss [1] in his original definition of definite self-adjointness.

THEOREM 9:1. *If the hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_6) *are satisfied and if a continuous* $f(x)$ *satisfies the relation* (5:13) *for every characteristic solution of the definitely self-adjoint boundary value problem* (3:1), *then* $Bf \equiv 0$ *on* $a \leqq x \leqq b$ *and* $s^1[f] = 0$.

The theorem follows immediately from the proof of Theorem 5:3, since there it is shown that $W_0 = K[u_0] = 0$ and thus, with the help of equation (5:18) for $q = 0$, that $Bf \equiv 0$ on $a \leqq x \leqq b$ and $s^1[f] = 0$.

COROLLARY 1. *If* $K[y] > 0$ *for all non-identically vanishing continuous* $y(x)$, *then* $f \equiv 0$ *is the only vector satisfying equation* (5:13) *with all characteristic solutions of the boundary value problem.*

COROLLARY 2. *The only* $f(x)$ *which satisfies a system of the form* (8:1) *with a continuous* $g(x)$ *and also the relation* (5:13) *with every solution of the boundary value problem* (3:1) *is* $f(x) \equiv 0$.

THEOREM 9:2. *If the system* (3:1) *satisfies the hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_6) *it has an infinity of characteristic values.*

Suppose that there are only a finite number of characteristic values, and denote by χ the number of linearly independent characteristic solutions of the system (3:1). We may also suppose that $\lambda = 0$ is not a characteristic value of this system. Since the elements of $B(x)$ are not all identically zero, continuous functions $g_\sigma(x)$ $(\sigma = 1, \ldots, \chi + 1)$ can be determined so that $B(x)g_\sigma(x)$ form $\chi + 1$ linearly independent vectors on $a \leqq x \leqq b$. Let $f_\sigma(x)$ denote the corresponding solution of the system (8:1) with $g(x)$ replaced by $g_\sigma(x)$. Then for arbitrary constants m_σ the vectors $f(x) = f_\sigma(x)m_\sigma$, $g(x) = g_\sigma(x)m_\sigma$ satisfy (8:1). On the other hand, values m_σ

($\sigma = 1, \ldots, \chi+1$) not all zero can be found so that f satisfies the relation (5:13) with each of the χ characteristic solutions of (3:1). It would then follow by the above Corollary 2 that $f \equiv 0$ on $a \leqq x \leqq b$, and consequently $0 \equiv Bg \equiv (Bg_\sigma)m_\sigma$ on this interval. This, however, contradicts the above choice of the sets g_σ. Hence under the hypotheses of Theorem 9:2 the system (3:1) has infinitely many characteristic values.

10. A boundary value problem of the calculus of variations. As pointed out in the introduction to this paper, we shall now show that an important case of the boundary value problem associated with the calculus of variations which has been discussed by Reid [6] is definitely self-adjoint according to the definition of Section 5. In the following section the symbols $\eta = (\eta_i)$, $\eta' = (\eta_i')$ will denote the functions $(\eta_1(x), \eta_2(x), \ldots, \eta_n(x))$ and the set of their derivatives respectively. The tensor analysis summation convention will be used throughout.

The second variation of the problem of Bolza is of the form

$$(10:1) \qquad J_2[\eta] = 2\mathcal{H}[\eta(x_1), \eta(x_2)] + \int_a^b 2\omega(x, \eta, \eta')\, dx,$$

where $2\mathcal{H}[\eta(x_1), \eta(x_2)]$ is a quadratic form in the 2n arguments $\eta_i(x_1)$, $\eta_i(x_2)$ and

$$2\omega(x, \eta, \eta') \equiv \omega_{\eta_i'\eta_j'}(x)\eta_i'\eta_j' + 2\omega_{\eta_i'\eta_j}(x)\eta_i'\eta_j + \omega_{\eta_i\eta_j}(x)\eta_i\eta_j$$

is a quadratic form in the 2n variables η_i, η_i'.

An accessory minimum problem associated with the second variation is that of minimizing $J_2[\eta]$ in a class of arcs

$$\eta_i = \eta_i(x) \qquad\qquad x_1 \leqq x \leqq x_2$$

satisfying conditions of the form

$$(10{:}2) \qquad \Phi_\alpha(x,\eta,\eta') \equiv \Phi_{\alpha\eta_j'}(x)\eta_j' + \Phi_{\alpha\eta_j}(x)\eta_j = 0 \qquad (\alpha = 1,2,\dots, m<n)$$

$$(10{:}3) \qquad \Psi_\gamma[\eta(x_1),\eta(x_2)] \equiv \Psi^1_{\gamma j}\,\eta_j(x_1) + \Psi^2_{\gamma j}\,\eta_j(x_2) = 0 \qquad (\gamma = 1,2,\dots, p \leqq 2n)$$

and giving a fixed value to the expression

$$(10{:}4) \qquad 2\mathcal{G}[\eta(x_1),\eta(x_2)] + \int_{x_1}^{x_2}\eta_i \mathcal{K}_{ij}\eta_j \, dx.$$

In (10:4), $2\mathcal{G}[\eta(x_1),\eta(x_2)]$ is a quadratic form in the 2n arguments $\eta_i(x_1)$, $\eta_i(x_2)$. For brevity, derivatives of the quadratic forms $\mathcal{G}$ and $\mathcal{H}$ with respect to the arguments $\eta_i(x_1)$, $\eta_i(x_2)$ will be indicated by $\mathcal{G}_{i1}, \mathcal{G}_{i2}, \mathcal{H}_{i1}, \mathcal{H}_{i2}$ respectively.

This accessory minimum problem is considered under the following hypotheses:

(I) *The coefficients in (10:1), (10:2), (10:3) and (10:4) are all real. The functions $\omega_{\eta_i'\eta_j'}$, $\omega_{\eta_i'\eta_j}$, $\Phi_{\alpha\eta_j'}$ are continuous and have continuous first derivatives, while the functions $\omega_{\eta_i\eta_j}$, $\Phi_{\alpha\eta_j}$, $\mathcal{K}_{ij}$ are continuous on x_1x_2. The matrices $\|\omega_{\eta_i'\eta_j'}\|$, $\|\omega_{\eta_i\eta_j}\|$ and $\|\mathcal{K}_{ij}\|$ are symmetric and $\|\Phi_{\alpha\eta_j'}\|$ is of rank m at each point on this interval. Moreover, the matrix of constants $\|\Psi^1_{\gamma j};\Psi^2_{\gamma j}\|$ has rank p. We also assume, as we may without loss of generality, that the rows of this latter matrix are normed and mutually orthogonal.*

(II) *The symmetric matrix*

$$(10{:}5) \qquad \begin{Vmatrix} \omega_{\eta_i'\eta_j'} & \Phi_{\beta\eta_i'} \\ \Phi_{\alpha\eta_j'} & 0_{\alpha\beta} \end{Vmatrix} \qquad (i,j=1,2,\dots,n;\ \alpha,\beta=1,\dots,m)$$

is non-singular on x_1x_2.

An arc $\eta = [\eta_i(x)]$ will be called differentially admissible if the functions $\eta_i(x)$ are continuous and have piece-wise continuous derivatives, and satisfy equations (10:2) on x_1x_2. An arc whose end values satisfy equations (10:3) will be called terminally admissible. Finally, an arc which is both differentially and terminally admissible will, for brevity, be called admissible.

(III). There exist p differentially admissible arcs $\eta_i = \eta_{i\nu}(x)$ $(\nu = 1,2, \ldots, p)$ such that the determinant $|\Psi_\gamma[\eta_\nu(x_1), \eta_\nu(x_2)]|$ is different from zero.

(IV). The form $2\mathcal{G}[\eta(x_1), \eta(x_2)]$ is positive semi-definite for all terminally admissible arcs η, and for x on x_1x_2 the quadratic form $\eta_i\mathcal{K}_{ij}\eta_j$ is positive definite.

By defining

$$(10{:}6)\qquad \Omega(x, \eta, \eta', \mu) = \omega(x, \eta, \eta') + \mu_\alpha \Phi_\alpha(x, \eta, \eta'),$$

the Euler-Lagrange differential equations and transversality conditions for the above defined accessory minimum problem can be written as

$$(10{:}7)\qquad \frac{d}{dx}\Omega_{\eta_i'} = \Omega_{\eta_i} - \lambda\mathcal{K}_{ij}\eta_j, \qquad \Phi_\alpha = 0,$$

$$a^1_{ri}\left\{\mathcal{H}_{i1}[\eta] - \lambda\mathcal{G}_{i1}[\eta] - \Omega_{\eta_i'}(x_1)\right\} + a^2_{ri}\left\{\mathcal{H}_{i2}[\eta] - \lambda\mathcal{G}_{i2}[\eta] + \Omega_{\eta_i'}(x_2)\right\} =$$

$$(10{:}8)\qquad \qquad (r = 1,2, \ldots, 2n-p),$$

$$\Psi_\gamma[\eta(x_1), \eta(x_2)] = 0 \qquad (\gamma = 1,2, \ldots, p),$$

where $a_r \equiv (a^1_{rj}, a^2_{rj})$ are $2n-p$ linearly independent solutions of the equations

$$\Psi^1_{\gamma j}a^1_j + \Psi^2_{\gamma j}a^2_j = 0.$$

Without any loss of generality we may assume that the solutions a_r are normed and mutually orthogonal. Consequently, the $2n \times 2n$ matrix

$$\begin{Vmatrix} \Psi^1_{\gamma j} & \Psi^2_{\gamma j} \\ a^1_{rj} & a^2_{rj} \end{Vmatrix}$$

is orthogonal and we have therefore

$$\begin{aligned} a^1_{rj}a^1_{ri} + \Psi^1_{\gamma j}\Psi^1_{\gamma i} &= I_{ji}, \\ a^1_{rj}a^2_{ri} + \Psi^1_{\gamma j}\,\Psi^2_{\gamma i} &= 0_{ji}, \\ a^2_{rj}a^1_{ri} + \Psi^2_{\gamma j}\,\Psi^1_{\gamma i} &= 0_{ji}, \\ a^2_{rj}a^2_{ri} + \Psi^2_{\gamma j}\,\Psi^2_{\gamma i} &= I_{ji}. \end{aligned} \tag{10:9}$$

The non-singularity of the matrix (10:5) insures that the system

$$\Phi_\alpha(x,\eta,\eta') = 0, \qquad \Omega_{\eta'_i}(x,\eta,\eta',\mu) = \zeta_i$$

has solutions

$$\eta'_i = \Pi_i(x,\eta,\zeta), \qquad \mu_\alpha = U_\alpha(x,\eta,\zeta)$$

which are linear in the variables η_i, ζ_i and with coefficients which are continuous functions of x on x_1x_2. Consequently, the differential equations (10:7) take the canonical form

$$\begin{aligned} \eta'_i &= \mathcal{A}_{ij}(x)\eta_j + \mathcal{B}_{ij}(x)\zeta_j, \\ \zeta'_i &= \mathcal{C}_{ij}(x)\eta_j - \mathcal{A}_{ji}(x)\zeta_j - \lambda\mathcal{K}_{ij}(x)\eta_j. \end{aligned} \tag{10:10}$$

The coefficients of (10:10) are continuous, the matrices $\|\mathcal{B}_{ij}\|$, $\|\mathcal{C}_{ij}\|$ are symmetric, and $\|\mathcal{B}_{ij}\|$ has rank $n-m$ on the interval x_1x_2. The boundary value problem consisting of (10:10) and the linear boundary conditions (10:8) has been shown to be self-adjoint

([5],[6]) under the transformation matrix

$$T = \left\| \begin{matrix} 0_{ij} & I_{ij} \\ -I_{ij} & 0_{ij} \end{matrix} \right\|$$

The corresponding matrix S is therefore

$$S = \left\| \begin{matrix} \mathcal{K}_{ij} & 0_{ij} \\ 0_{ij} & 0_{ij} \end{matrix} \right\|$$

which is symmetric since the matrix $\|\mathcal{K}_{ij}\|$ is symmetric. Conditions (H_1) and (H_2) of the definition of definitely self-adjoint problems are thus satisfied for the system (10:10), (10:8).

Consider again the boundary conditions (10:8). By a property of quadratic forms ([2], pp. 11-12) this system can be written as

$$\begin{aligned} &\{\mathcal{H}_{i1}[a_r] - \lambda\mathcal{G}_{i1}[a_r]\}\eta_i(x_1) - a^1_{ri}\zeta_i(x_1) \\ (10{:}11) \quad &+ \{\mathcal{H}_{i2}[a_r] - \lambda\mathcal{G}_{i2}[a_r]\}\eta_i(x_2) + a^2_{ri}\zeta_i(x_2) = 0, \\ &\Psi_\gamma\ [\eta(x_1), \eta(x_2)] = 0 \qquad (\gamma = 1,\dots,p; r = 1,2,\dots,2n-p). \end{aligned}$$

It is seen that the above boundary conditions are of the same form as those in the system (3:1) for the sets η_i, ζ_i, but the matrices $M(\lambda)$, $N(\lambda)$ are now 2n-dimensional. We will now transform these boundary conditions to a more convenient form. Multiply the matrix of coefficients of the equation (10:11) on the left by the non-singular matrix

$$(10{:}12) \qquad \left\| \begin{matrix} I_{rs} & \lambda\mathcal{G}\ [a_r; \Psi_\gamma] \\ 0_{\gamma s} & I_{\gamma\nu} \end{matrix} \right\| \qquad (\gamma, \nu = 1,2,\dots,p; r,s = 1,\dots,2n-p).$$

In (10:12) it is to be understood that $\mathcal{G}\,[a_r;\Psi_\gamma]$ is the bilinear expression $\mathcal{G}_{i1}[a_r]\Psi^1_{\gamma i} + \mathcal{G}_{i2}[a_r]\Psi^2_{\gamma i}$. Then we obtain for the matrices M(λ) and N(λ) the following expressions, in which it is to be understood that i, r, and γ are row indices and j, s, and ν are column indices, and the range of these indices are as indicated previously.

$$M(\lambda) = \begin{Vmatrix} \mathcal{H}_{j1}[a_r] - \lambda\,\mathcal{G}_{j1}[a_r] + \lambda\,\mathcal{G}\,[a_r;\Psi_\nu]\Psi^1_{\nu j} & -a^1_{rj} \\ \\ \Psi^1_{\gamma j} & 0_{\gamma j} \end{Vmatrix},$$

$$N(\lambda) = \begin{Vmatrix} \mathcal{H}_{j2}[a_r] - \lambda\,\mathcal{G}_{j2}[a_r] + \lambda\,\mathcal{G}\,[a_r;\Psi_\nu]\Psi^2_{\nu j} & a^2_{rj} \\ \\ \Psi^2_{\gamma j} & 0_{\gamma j} \end{Vmatrix}.$$

The transformation (10:12) obviously does not change the homogeneous boundary conditions (10:11).

An application of equations (2:2) and (4:2) give us expressions for P(λ) and Q(λ) as follows:

$$P(\lambda) = \begin{Vmatrix} a^1_{si} & 0_{i\nu} \\ \\ \mathcal{H}_{i1}[a_s] - \lambda\,\mathcal{G}_{i1}[a_s] + \lambda\,\mathcal{G}\,[a_s;\Psi_\gamma]\Psi^1_{\gamma i} & \Psi^1_{\nu i} \end{Vmatrix},$$

$$Q(\lambda) = \begin{Vmatrix} -a^2_{si} & 0_{i\nu} \\ \\ \mathcal{H}_{i2}[a_s] - \lambda\,\mathcal{G}_{i2}[a_s] + \lambda\,\mathcal{G}\,[a_s;\Psi_\gamma]\Psi^2_{\gamma i} & \Psi^2_{\nu i} \end{Vmatrix}.$$

With the help of equations (10:9) we can determine the matrices M^*, N^*, P^*, Q^* independent of λ in such a way that the matrices (2:1) are reciprocals. They can be chosen having the following form

$$P^* = \left\| \begin{matrix} 0_{is} & -\Psi^1_{\nu i} \\ a^1_{si} & -\mathcal{H}_{i1}[\Psi_\nu] \end{matrix} \right\|, \qquad Q^* = \left\| \begin{matrix} 0_{is} & \Psi^2_{\nu i} \\ a^2_{si} & -\mathcal{H}_{i2}[\Psi_\nu] \end{matrix} \right\|,$$

$$M^* = \left\| \begin{matrix} -a^1_{rj} & 0_{rj} \\ \mathcal{H}_{j1}[\Psi_\gamma] & -\Psi^1_{\gamma j} \end{matrix} \right\|, \qquad N^* = \left\| \begin{matrix} -a^2_{rj} & 0_{rj} \\ \mathcal{H}_{j2}[\Psi_\gamma] & \Psi^2_{\gamma j} \end{matrix} \right\|$$

A rather lengthy computation and use of equations (10:9) yield for the matrix of coefficients (5:1) of the form Q for this special problem the matrix

$$(10:13) \quad \left\| \begin{matrix} a^1_{ri}\mathcal{G}[a_r;a_s]a^1_{sj} & 0_{ij} & a^1_{ri}\mathcal{G}[a_r;a_s]a^2_{sj} & 0_{ij} \\ 0_{ij} & 0_{ij} & 0_{ij} & 0_{ij} \\ a^2_{ri}\mathcal{G}[a_r;a_s]a^1_{sj} & 0_{ij} & a^2_{ri}\mathcal{G}[a_r;a_s]a^2_{sj} & 0_{ij} \\ 0_{ij} & 0_{ij} & 0_{ij} & 0_{ij} \end{matrix} \right\|$$

where $\mathcal{G}[a_r;a_s] = \mathcal{G}[a_s;a_r] = a^1_{ri}\mathcal{G}_{i1}[a_s] + a^2_{ri}\mathcal{G}_{i2}[a_s]$. The matrix (10:13) is seen to be symmetric; that is, hypothesis (H_3) of the definition of definite self-adjointness is satisfied.

Corresponding to arbitrary values $\eta_i(x_1)$, $\eta_i(x_2)$ the equations

$$\eta_i(x_1) = c_r a^1_{ri} + d_\gamma \Psi^1_{\gamma i}$$

$$\eta_i(x_2) = c_r a^2_{ri} + d_\gamma \Psi^2_{\gamma i}$$

have the unique solution

$$c_r = a^1_{ri}\,\eta_i(x_1) + a^2_{ri}\,\eta_i(x_2),$$

$$d_\gamma = \Psi^1_{\gamma i}\,\eta_i(x_1) + \Psi^2_{\gamma i}\,\eta_i(x_2) \qquad \begin{array}{l}(\gamma = 1,\dots,p; \\ \;r = 1,2,\dots,2n-p).\end{array}$$

In particular, for a terminally admissible arc the constants d_γ are zero. For general sets $\eta_i(x_1)$, $\zeta_i(x_1)$, $\eta_i(x_2)$, $\zeta_i(x_2)$ the form Q of (H_3) for this problem turns out to be the quadratic form $2\mathcal{G}$ with the corresponding arguments $c_r a^1_{ri}$, $c_r a^2_{ri}$. As $\Psi^1_{\gamma j}\,a^1_{rj} + \Psi^2_{\gamma j}\,a^2_{rj} = 0$ $(\gamma = 1,\dots,p;\ r = 1,2,\dots,2n-p)$, hypothesis (IV) is seen to imply condition (H_4) for this boundary value problem. Finally, the above hypothesis (III) may be shown to be equivalent to the condition that there is no characteristic solution η_i, ζ_i of this boundary value problem for which $\eta_i \equiv 0$ on x_1x_2 (see, for example Bliss [3], p. 48). Consequently hypothesis (H_5) is also satisfied and the above defined accessory boundary value problem has been shown to be definitely self-adjoint according to the definition of Section 5.

The above hypotheses on the boundary value problem considered in this section are more special than those of Reid [6], in that we have assumed the positiveness of the quadratic form $G[\eta(a),\eta(b)]$ and the positive definiteness of $\eta_i K_{ij}\,\eta_j$. On the other hand, in another sense our conditions are weaker in that we have only assumed the non-singularity of the matrix (10:5), whereas the hypotheses of Reid imply that $\omega_{\eta_i'\eta_j'}(x)\pi_i\pi_j > 0$ for all sets $(\pi_j) \neq (0_j)$ satisfying $\Phi_{\alpha\eta_j'}(x)\pi_j = 0$ $(\alpha = 1, \dots, m)$.

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APPLICATIONS OF THE COVARIANT DERIVATIVE OF CARTAN
IN THE
CALCULUS OF VARIATIONS

BY

ELLEN CLAYTON STOKES

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APPLICATIONS OF THE COVARIANT DERIVATIVE OF CARTAN IN THE CALCULUS OF VARIATIONS

Introduction. The purpose of this paper is to express in tensor form the criteria for a minimizing arc for a parametric problem of the calculus of variations with fixed end-points. The problem to be considered is in the n-dimensional space of points $y = (y^1, \ldots, y^n)$ and in the form developed by Wren [8]*. It is that of finding in a class of admissible arcs one which minimizes an integral of the form

$$I = \int_{t_1}^{t_2} [2g(y^1, \ldots, y^n, y^{1\prime}, \ldots, y^{n\prime})]^{1/2} dt = \int_{t_1}^{t_2} [2g(y,y')]^{1/2} dt,$$

the function $g(y,y')$ being positively homogeneous of order two in the variables y'.

For a tensor treatment of functions which involve derivatives it is desirable to have a definition of an absolute differential and of related covariant derivatives. Several different covariant derivatives have been suggested in the literature, notably those of Ricci [1] for Riemannian spaces, and of Taylor [6], Berwald [7], and Cartan [9] for Finsler spaces [4]. Of these that of Cartan has been chosen for use in this paper since it includes the others as special cases and seems to be the most general.

In Section 1 a formulation of Wren's problem is given. The condition is imposed that the integrand function $(2g)^{1/2}$ be greater than zero in a certain region R of yy'-space, but a proof is presented to show that this is not a real restriction, at least when R is a restricted neighborhood of a given curve C.

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*Numbers in square brackets refer to the bibliography at the end of the paper.

In the next two sections the fundamental tensors to be employed are introduced and properties characterizing the absolute differential of Cartan are given. These properties differ somewhat from those which Cartan himself gave. He developed his differential primarily for the study of Finsler spaces. In the present paper the definition is related more directly to problems of the calculus of variations.

In Section 4 Wren's first variation and his Euler differential equations have been transformed to a tensor form involving Cartan's covariant derivative. There, as in Sections 6 and 7 where the second variation and the Jacobi differential equations are discussed, the results agree with those of Householder [10] for the variable end-point problem. Section 5 deals with the invariant properties of the Weierstrass E-function and the Legendre quadratic form. In Section 8 we have found the transformation formula for the determinant whose zeros determine the conjugate points of the minimizing arc. Finally in Section 9 the tensor properties of the derivatives of the Hamiltonian function are presented, and the canonical differential equations are expressed in terms of covariant derivatives.

Throughout the paper the summation convention of tensor analysis has been freely used, namely, that in a product a repeated index indicates a summation. For example, $\ell_i y^i$ stands for $\sum_{i=1}^{n} \ell_i y^i$.

1. *The integral function and its properties*. We first present a well-known form of the parametric problem of the calculus of variations with fixed end-points. The integrand considered in n-dimensional space is

$$(1:1) \qquad I = \int_{t_1}^{t_2} f(y^1, \ldots, y^n, y^{1\prime}, \ldots, y^{n\prime})dt = \int_{t_1}^{t_2} f(y,y')dt$$

and the arcs C along which it is to be evaluated are defined by functions of the form

$$(1:2) \qquad y^i(t) \qquad (t_1 \leqq t \leqq t_2;\ i = 1,\ldots,n).$$

We suppose that there is a region R in the 2n-dimensional space of points (y,y') with the following properties:

(a) $y^{i\prime}y^{i\prime} \neq 0$ at all points (y,y') in R,

(b) if (y,y') is in R then (y,ky') is also in R for every $k > 0$,

(c) $f(y,y')$ satisfies the homogeneity relation $f(y,ky') = kf(y,y')$ for every $k > 0$ and (y,y') in R,

(d) $f(y,y')$ has continuous partial derivatives of at least the fourth order in R,

(e) $f(y,y') > 0$ for (y,y') in R,

(f) the matrix $\| \partial^2 f/\partial y^{i\prime}\partial y^{j\prime} \|$ is of rank $n-1$ for (y,y') in R.

This last requirement makes the calculus of variations problem to be considered non-singular in R.

All elements (y,y') interior to R will be termed *admissible elements*. An *arc* defined by (1:2) will be said to be *regular* if the functions $y^i(t)$ possess continuous derivatives $y^{i\prime}$ not vanishing simultaneously on the t-interval belonging to the arc. An *admissible arc* is one whose elements (y,y') are admissible and which is continuous and made up of a finite number of regular sub-arcs on the interval t_1t_2.

The problem is that of finding in the class of admissible arcs joining two given fixed points 1 and 2 in the n-dimensional space of points $y = (y^1, \ldots, y^n)$ one which minimizes the integral (1:1).

We now transform this problem to the one studied by Wren [8] by expressing the integrand in terms of the function $g(y,y')$ defined by the equation

$$2g(y,y') = f^2(y,y'),$$

so that the integral (1:1) becomes

$$I = \int_{t_1}^{t_2} [2g(y,y')]^{1/2} dt. \tag{1:3}$$

The function $g(y,y')$ will possess in R all the properties of $f(y,y')$ with the exception that (c) and (f) are now, respectively, equivalent to

(c') $g(y,ky') = k^2 g(y,y')$ for every $k > 0$,

(f') $|\partial^2 g/\partial y^{i'} \partial y^{j'}| \neq 0$ in R.

The last statement can be verified easily by means of the definition of $g(y,y')$, the homogeneity property of $f(y,y')$, and well-known matrix theory.

We list here some of the consequences of property (c') which will be of use in the study which follows [8, p. 168]:

$$\begin{aligned} &g_i y^{i'} = 2g, && g_{ijk} y^{k'} = 0, \\ &g_{ij} y^{j'} = g_i, && g_{iy^k} y^{i'} = 2g_{y^k}, \\ &g_{ij} y^{i'} y^{j'} = 2g, && g_{ijy^k} y^{j'} = g_{iy^k}, \end{aligned} \tag{1:4}$$

where

$$g_i = \partial g/\partial y^{i'}, \qquad g_{ij} = \partial^2 g/\partial y^{i'} \partial y^{j'}, \qquad g_{y^k} = \partial g/\partial y^k.$$

These notations will be used throughout the paper.

The condition (e) that $f(y,y') > 0$ in R is not a real restriction upon the nature of the calculus of variations problem described above, at least when the region R is a restricted neighborhood of a given curve C of class C^v[2, p. 13]. For suppose $f(y,y')$ not always positive on the arc C but that R and f otherwise satisfy the conditions (a) - (f). As will be shown below, there exists a function $\varphi(y)$, having continuous fifth derivatives, whose derivative $d\varphi/dt$ will be positive along the given arc C and in a restricted neighborhood of the elements (y,y') on C. The integrand $f(y,y')$ may then be replaced in the integral I of equation (1:1) by $F(y,y') = f(y,y') + k^2 d\varphi/dt$, which for a sufficiently large value of k will be positive along C since $d\varphi/dt$ is positive on C. This function $F(y,y')$ satisfies all the properties (a) - (f) in a neighborhood R of the elements (y,y') on C, as one may readily verify. Moreover, such a change of integrand function does not affect the extremals for I, since the Euler differential equations are identically zero for a derivative function $d\varphi/dt$[2, p. 153].

We now proceed to prove the existence of the function $\varphi(y)$. Suppose that the arc C joining the points 1 and 2 is defined by functions

$$y^i(\xi) \qquad (i = 1,\dots,n)$$

of class C^v, where ξ is the length of arc measured along C from the point 1. Suppose further that C does not intersect itself between the points 1 and 2 which correspond to $\xi = 0$ and $\xi = \xi_s$, respectively. The derivatives $y^{i\prime}(\xi)$ are the components of the vector tangent to C. Let

$$v_1^i(\xi), \ldots, v_{n-1}^i(\xi) \qquad (i = 1, \ldots, n)$$

be the components of class C^v of $n-1$ vectors which with the tangent vector form a set of n mutually orthogonal vectors at the point ξ on the curve, so that the determinant

$$(1{:}5) \qquad \begin{vmatrix} y^{1\prime}(\xi) & v_1^1(\xi) & \ldots & v_{n-1}^1(\xi) \\ \ldots & \ldots & \ldots & \ldots \\ y^{n\prime}(\xi) & v_1^n(\xi) & \ldots & v_{n-1}^n(\xi) \end{vmatrix}$$

is different from zero on the arc C. The equations

$$(1{:}6) \qquad y^i = y^i(\xi) + \zeta_j v_j^i(\xi) \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n-1)$$

define a transformation of coordinates from $(y^1, \ldots, y^n)$ to $(\xi, \zeta_1, \ldots, \zeta_{n-1})$. It is apparent that to every set (ξ, ζ) there corresponds one set (y).

Let us consider the solution of these equations for ξ and the ζ_i's in terms of the y^i's. We know that there is one set of solutions

$$(y^1, \ldots, y^n, \xi, \zeta_1, \ldots, \zeta_{n-1}) = [y^1(\xi), \ldots, y^n(\xi), \ ,0, \ldots, 0]$$

belonging to the arc C. Moreover, no two distinct solutions of this set can have the same projection in the y-space, since by hypothesis the curve does not intersect itself. In addition, the fact that the functional determinant of equations (1:6) with respect to $\xi, \zeta_1, \ldots, \zeta_{n-1}$ is different from zero on the solutions given above follows from the non-vanishing of the determinant (1:5) Therefore, by implicit function theory [3, pp. 8-11] the equations (1:6) have unique solutions in a restricted neighborhood of the curve C and we may write

$$
\begin{aligned}
\xi &= \varphi(y^1, \ldots, y^n) = \varphi(y) \\
\zeta_i &= \psi_i(y^1, \ldots, y^n) \qquad (i = 1, \ldots n-1).
\end{aligned}
\tag{1:7}
$$

These functions will be of class at least C^v if the arc C is of class C^v.

Now in the ξ, ζ space, the curve C becomes the segment $O\xi_2$ on the ξ-axis, and we know that along this segment ξ itself increases monotonically from 0 to ξ_2. Therefore $\varphi(y)$ is a function whose derivative is positive along C and, by continuity, at least in a small neighborhood of the elements (y, y') on C.

2. Fundamental tensors. The type of transformation of coordinates which is to be considered in this paper is defined by equations

$$\bar{y}^i = \bar{y}^i(y^1, \ldots, y^n) \qquad (i = 1, \ldots, n) \tag{2:1}$$

where $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^n)$ is the set of new coordinates. It is supposed that the equations (2:1) are of class C^v in the projection S in y-space of the region R, and that they define with equations

$$\bar{y}^{i\prime} = y^{k\prime}\partial \bar{y}^i / \partial y^k \tag{2:2}$$

a one to one correspondence between the region R and a region $\bar{R}$ of points $(\bar{y}, \bar{y}')$. Furthermore we assume that the determinant $|\partial \bar{y}^i / \partial y^k|$ is different from zero in R. According to well-known implicit function theory the inverse of the transformation (2:1), namely,

$$y^i = y^i(\bar{y}^1, \ldots, \bar{y}^n) \qquad (i = 1, \ldots, n)$$

is then also of class C^v in $\bar{R}$.

Sets of values Y^i, Y_i associated with the point y which are transformed into sets $\bar{Y}^i$, $\bar{Y}_i$ at the point $\bar{y}$ by the transformations

$$(2:3) \qquad \bar{Y}^i = Y^k \partial \bar{y}^i/\partial y^k, \qquad \bar{Y}_i = Y_k \partial y^k/\partial \bar{y}^i$$

are called, respectively, contravariant and covariant tensors of rank one. It follows readily by differentiating (2:1) that the derivatives $y^{i\prime}$ associated with a curve $y^i(t)$ satisfy (2:2) and are the components of a contravariant tensor.

We assume that the function $g(y,y')$ is an invariant so that by definition

$$\bar{g}(\bar{y},\bar{y}') = g(y,y').$$

It then follows by differentiating this equation that g_i, g_{ij}, and g_{ijk} are the components of covariant tensors of ranks one, two, and three, respectively. We define g^{ij} to be the element corresponding to g_{ij} in the reciprocal of the determinant $|g_{ij}|$, so that $g^{ij}g_{ik} = \delta_k^{\ j}$ where $\delta_k^{\ j}$ is 1 if $j = k$ and 0 if $j \neq k$. Then g^{ij} is also a tensor of rank 2 [5, p. 43]. The tensors g_{ij} and g^{ij} are often used for lowering and raising indices. The tensors ℓ^i and $\ell_i = g_{ij}\ell^j$ are called corresponding contravariant and covariant tensors.

We call attention to the fact that g_{y^k} is not a tensor, as is seen from the formula

$$\begin{aligned} \bar{g}_{\bar{y}^k} &= g_{y^r}\partial y^r/\partial \bar{y}^k + g_s \partial y^{s\prime}/\partial \bar{y}^k \\ &= g_{y^r}\partial y^r/\partial \bar{y}^k + g_s(\partial^2 y^s/\partial \bar{y}^k \partial \bar{y}^h) y^{r\prime} \partial \bar{y}^h/\partial y^r. \end{aligned}$$

As usual we define the length of a vector of components Y^i to be $(g_{ij}Y^iY^j)^{1/2}$, and the cosine of the angle θ between the vectors of components Y^i and X^j to be

$$\cos\theta = g_{ij}X^iY^j/(g_{hk}X^hX^k)^{1/2}(g_{rs}Y^rY^s)^{1/2}.$$

From the definition of vector length just given it follows that the numbers

$$(2:4) \qquad \ell^i = y^{i'}/(g_{kj}y^{k'}y^{j'})^{1/2} = y^{i'}/(2g)^{1/2}$$

are the components of a contravariant unit vector in the direction defined by the $y^{i'}$, and that the numbers

$$(2:5) \qquad \ell_i = g_{ij}y^{j'}/(g_{hk}y^{h'}y^{k'})^{1/2} = g_i/(2g)^{1/2}$$

are the components of the corresponding covariant vector. Moreover

$$\ell^i\ell_i = g_{ij}y^{i'}y^{j'}/2g = 1.$$

3. *The absolute differential of Cartan.* The following postulates, somewhat different from those of Cartan [9, p. 10], are sufficient to define his absolute differential for which the symbol D will be used below.

(A) The absolute differential of an invariant is the ordinary differential.

(B) DY^i is a function of the variables y, y', dy, dy', Y, dY, linear in Y and dY so that

$$D(cY^i) = cDY^i, \qquad D(Y^i + Z^i) = DY^i + DZ^i.$$

From this property it follows that the elements DY^i have the form

$$DY^i = M^i_k\, dY^k + C^i_k Y^k.$$

(C) DX_i has similar properties relative to X_i and dX_i, so that

$$DX_i = L_i^k\, dX_k + B_i^k X_k.$$

(D) For all outer and inner products of tensors the usual distributive law of the calculus holds, for example

$$D(X_iY^i) = X_iDY^i + Y^iDX_i.$$

If Y^i is the tensor whose components in y-space are δ^i_j for a fixed j, it follows from the fact that $D(X_iY^i) = d(X_iY^i)$ that the matrices L_i^k and M_k^i are the identity matrices and the equations

$$DX_i = dX_i - C_i^k X_k,$$

$$DY^i = dY^i + C_k^i Y^k$$

are valid.

(E) The elements of DY^i are polynomials in dy^i and $dy^{i\prime}$ and are linear and homogeneous of order one in these arguments. It then follows that we may write

$$\begin{aligned} DY^i &= dY^i + Y^k(\Gamma_k{}^i{}_h dy^h + C_k{}^i{}_h dy^{h\prime}), \\ DX_i &= dX_i + X_k(\Gamma_i{}^k{}_h dy^h + C_i{}^k{}_h dy^{h\prime}). \end{aligned} \tag{3:1}$$

(F) $g_{ij}Y^iZ^j$ is constant when $DY^i = DZ^i = 0$.

(G) $C_{ijh} = C_{jih}$.

From the property (F) it follows that

$$d(g_{ij}Y^iZ^j) = Y^iZ^j dg_{ij} + g_{ij}Z^j dY^i + g_{ij}Y^i dZ^j = 0$$

when $DY^i = DZ^j = 0$, so that from (3:1)

$$dg_{ij} = (\Gamma_{ijh} + \Gamma_{jih})dy^h + (C_{ijh} + C_{jih})dy^{h'},$$

and hence

(3:2)
$$g_{ij\,y^h} = \Gamma_{ijh} + \Gamma_{jih},$$
$$g_{ijh} = C_{ijh} + C_{jih}.$$

Then from property (G) we have

(3:3)
$$C_{ijh} = \frac{1}{2}\, g_{ijh}$$

and therefore C_{ijh} is symmetric in all three indices.

It follows now from equations (1:4) that $C_{ijh}y^{i'} = 0$ and hence from (3:1) that $Dy^{i'}$ takes the form

(3:4)
$$Dy^{i'} = dy^{i'} + y^{k'}\Gamma_{k}{}^{i}{}_{h}dy^h.$$

With the aid of this equation the equations (3:1) may be written in the form

(3:5)
$$DY^i = dY^i + Y^k(\Gamma^{*}{}_{k}{}^{i}{}_{h}dy^h + C_{k}{}^{i}{}_{h}Dy^{h'}),$$
$$DX_i = dX_i - X_k(\Gamma^{*}{}_{i}{}^{k}{}_{h}dy^h + C_{i}{}^{k}{}_{h}Dy^{h'}),$$

where

(3:6)
$$\Gamma^{*}{}_{i}{}^{k}{}_{h} = \Gamma_{i}{}^{k}{}_{h} - C_{i}{}^{k}{}_{r}y^{m'}\Gamma_{m}{}^{r}{}_{n}.$$

The final property assumed for the absolute differential is the further symmetry property

(H) $\Gamma^{*}{}_{k}{}^{i}{}_{h} = \Gamma^{*}{}_{h}{}^{i}{}_{k}.$

From these postulates we may find, with Cartan [9, pp. 14-16], the following values of the coefficients Γ and Γ^{*}:

(3:7)
$$\Gamma_{kjh} = \gamma_{kjh} + C_{khr}\partial G^r/\partial y^{j'} - C_{jhr}\partial G^r/\partial y^{k'},$$
$$\Gamma^*_{kjh} = \gamma_{kjh} + C_{khr}\partial G^r/\partial y^{j'} - C_{kjr}\partial G^r/\partial y^{h'} - C_{jhr}\partial G^r/\partial y^{k'},$$

where

$$\gamma_{kjh} = (g_{kjy^h} + g_{jhy^k} - g_{khy^j})/2,$$
$$G_m = (g_{my^k}y^{k'} - g_{y^m})/2.$$

As we shall see later, the Euler differential equations of Wren take the form

$$dg_i/dt - g_{y^i} = g_{ij}y^{j''} + g_{iy^j}y^{j'} - g_{y^i} = 0.$$

By means of the value of G_m given above these equations become

$$g_{ij}y^{j''} + 2G_i = 0,$$

which, upon contraction with g^{ik}, are seen to be equivalent to

$$y^{k''} + 2G^k = 0.$$

We also make note of Cartan's equation X [9, p. 16] which will be of use to us later

(3:8)
$$y^{h'}\Gamma_h{}^i{}_j = \partial G^i/\partial y^{j'}.$$

The first member of this equation is equal to $y^{h'}\Gamma^{*}{}_h{}^i{}_j$, as may readily be verified by means of the relations (1:4).

A most important result of the above definition is the fact that the absolute differential of a covariant or of a contravariant tensor is itself a tensor of the same kind. This can be shown by a transformation of coordinates of the kind described in Section 2. The proof is too long to be reproduced here in detail, so that we shall give only the main steps in the argument.

Suppose that a bar over a function indicates that the arguments are $\bar{y}$ and $\bar{y}'$. We wish to prove that $\overline{DY}^i = DY^\alpha \partial \bar{y}^i/\partial y^\alpha$. Let us consider the laws of transformation of the elements of

$$\overline{DY}^i = d\bar{Y}^i + \bar{Y}^k(\bar{\Gamma}_k{}^i{}_h d\bar{y}^h + \bar{C}_k{}^i{}_h d\bar{y}^{h\prime}). \tag{3:9}$$

Differentiating the expression for Y^i in equation (2:3) we find that

$$d\bar{Y}^i = dY^\alpha \partial \bar{y}^i/\partial y^\alpha + Y^\alpha dy^\beta \partial^2 \bar{y}^i/\partial y^\alpha \partial y^\beta . \tag{3:10}$$

Since $\bar{y}^{h\prime}$ is a contravariant tensor, $d\bar{y}^{h\prime}$ transforms by the same formula (3:10). The coefficient $\bar{C}_k{}^i{}_h$ is a mixed tensor since it is the inner product of the tensor g^{ij} by the tensor C_{kjh}, and $d\bar{y}^h$ is also a tensor. Furthermore we find after a somewhat extended computation that the elements of $\bar{\Gamma}_k{}^i{}_h$ transform as follows:

$$\bar{G}_r = (1/2)[g_{\beta y\alpha}\, y^{\alpha\prime} \partial y^\beta/\partial \bar{y}^r$$

$$+ g_{\beta\alpha}\, y^{\sigma\prime}(\partial y^\beta/\partial \bar{y}^r)(\partial \bar{y}^s/\partial y^\sigma)(\partial y^{\alpha\prime}/\partial \bar{y}^s) - g_{y\alpha}\partial y^{\alpha\prime}/\partial \bar{y}^r],$$

$$\bar{C}_{khr}\partial \bar{G}^r/\partial \bar{y}^{j\prime} = C_{\kappa\eta\rho}(\partial G^\rho/\partial y^{\nu\prime})(\partial y^\kappa/\partial \bar{y}^k)(\partial y^\eta/\partial \bar{y}^h)(\partial y^\nu/\partial \bar{y}^j)$$

$$+ C_{\kappa\alpha\eta}(\partial y^\kappa/\partial \bar{y}^k)(\partial y^\eta/\partial \bar{y}^h)(\partial y^{\alpha\prime}/\partial \bar{y}^j),$$

$$\bar{\gamma}_{kjh} = \gamma_{\kappa\nu\eta}(\partial y^\kappa/\partial \bar{y}^k)(\partial y^\nu/\partial \bar{y}^j)(\partial y^\eta/\partial \bar{y}^h) \tag{3:11}$$

$$+ g_{\kappa\nu}(\partial y^\nu/\partial \bar{y}^j)(\partial^2 y^\kappa/\partial \bar{y}^k \partial \bar{y}^h)$$

$$+ C_{\kappa\nu\eta}[(\partial y^\kappa/\partial \bar{y}^k)(\partial y^\nu/\partial \bar{y}^j)(\partial y^{\eta\prime}/\partial \bar{y}^h)$$

$$+ (\partial y^{\kappa\prime}/\partial \bar{y}^k)(\partial y^\nu/\partial \bar{y}^j)(\partial y^\eta/\partial \bar{y}^h)$$

$$- (\partial y^\kappa/\partial \bar{y}^k)(\partial y^{\nu\prime}/\partial \bar{y}^j)(\partial y^\eta/\partial \bar{y}^h)].$$

Then an application of these equations (3:11) and the statements preceding them to the first of equations (3:7) shows that

$$
\begin{aligned}
\Gamma_{k\ h}^{\ i} = {} & \Gamma_{\kappa\ \eta}^{\ \mu}(\partial y^{\kappa}/\partial \bar{y}^{k})(\partial y^{\eta}/\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu}) + (\partial^{2} y^{\mu}/\partial \bar{y}^{k}\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu}) \\
& + C_{\kappa\ \eta}^{\ \mu}(\partial y^{\kappa}/\partial \bar{y}^{k})(\partial y^{\eta'}/\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu}).
\end{aligned}
\tag{3:12}
$$

Finally, with the aid of the above laws of transformation, we see that

$$\overline{DY^{i}} = (\partial \bar{y}^{i}/\partial y^{\mu})[dY^{\mu} + Y^{\kappa}(\Gamma_{\kappa\ \eta}^{\ \mu}\, dy^{\eta} + C_{\kappa\ \eta}^{\ \mu}\, dy^{\eta'})],$$

which proves the tensor property. By these same formulas it also follows that DX_i is a covariant tensor.

The definition of the absolute differential can be extended so that it will apply to tensors of higher order than the first. In order to make this extension let us first consider the special covariant tensor $T_{ij} = X_i Y_j$. Then by postulate (D) and equations (3:1) we have

$$
\begin{aligned}
DT_{ij} &= X_i DY_j + Y_j DX_i, \\
&= X_i dY_j + Y_j dX_i - X_i Y_k \Gamma_{j\ h}^{\ k} dy^{h} - Y_j X_k \Gamma_{i\ h}^{\ k} dy^{h} \\
&\quad - X_i Y_k C_{j\ h}^{\ k} dy^{h'} - Y_j X_k C_{i\ h}^{\ k} dy^{h'} \\
&= dT_{ij} - T_{ik}\Gamma_{j\ h}^{\ k} dy^{h} - T_{kj}\Gamma_{i\ h}^{\ k} dy^{h} - T_{ik} C_{j\ h}^{\ k} dy^{h'} \\
&\quad - T_{kj} C_{i\ h}^{\ k} dy^{h'}.
\end{aligned}
\tag{3:13}
$$

This will be taken as the definition of DT_{ij} for an arbitrary tensor T_{ij}. It can readily be shown to be itself a covariant tensor of the second rank. In the same way we find that if $T_i^{\ j} = X_i Y^j$ then

$$DT_i{}^j = dT_i{}^j - T_k{}^j \Gamma_i{}^k{}_h dy^h + T_i{}^k \Gamma_k{}^j{}_h \, dy^h$$
$$- T_k{}^j C_i{}^k{}_h dy^h - T_i{}^k C_k{}^j{}_h \, dy^{h'},$$

and for a tensor T^{ij}

$$DT^{ij} = dT^{ij} + T^{ik}\Gamma_k{}^j{}_h \, dy^h + T^{kj}\Gamma_k{}^i{}_h \, dy^{h'} \qquad (3:14)$$
$$+ T^{ik}C_k{}^j{}_h \, dy^{h'} + T^{kj}C_k{}^i{}_h \, dy^{h'}.$$

Proceeding in this way we may find the form for the absolute differential of any tensor $T^{s_1 \ldots s_e}_{r_1 \ldots r_m}$ and the result will be a tensor of the same kind.

An immediate result of equations (3:2) and (3:13) is that

$$Dg_{ij} = 0 \qquad (3:15)$$

and similarly, with the aid of the equation $g^{ij}g_{jk} = \delta^i_k$,

$$Dg^{ij} = 0. \qquad (3:16)$$

4. The first variation and the Euler differential equations. In constructing the first and second variations Wren [8, p. 168] considers the one-parameter family of curves

$$y^i(t,a) = y^i(t) + a\eta^i(t) \qquad (t_1 \leqq t \leqq t_2; \; i = 1, \ldots, n), \qquad (4:1)$$

where the $\eta^i(t)$ are a set of admissible variations satisfying the conditions

$$\eta^i(t) \text{ of class } D' \text{ for } t_1 \leqq t \leqq t_2,$$
$$\eta^i(t_1) = \eta^i(t_2) = 0.$$

The family contains the curve C defined by functions (1:2) when $a = 0$. The η^i are the components of a contravariant tensor since

$$\eta^i = \partial[y^i(t,a)]/\partial a, \qquad \partial \bar{y}^i/\partial a = (\partial \bar{y}^i/\partial y^\alpha)(\partial y^\alpha/\partial a)$$

and hence

$$\eta^i(t) = \eta^\alpha \, \partial \bar{y}^i/\partial y^\alpha .$$

The first variation as found by Wren is

$$I'(0) = \int_{t_1}^{t_2} (g_{yi}\eta^i + g_i \eta^{i'})(2g)^{-1/2} dt. \tag{4:2}$$

Since the derivative of an invariant is an invariant, it is clear that the integrand function, which is the partial derivative of $(2g)^{1/2}$ with respect to a, is an invariant. However, we may express this integrand in terms of tensors in such a way that from its very form its invariance is at once apparent. This we proceed to do.

With the aid of the homogeneity relations (1:4) the expression for Γ^*_{hmi} in equation (3:7) provides the relationship

$$y^{h'}y^{m'}\Gamma^*_{hmi} = g_{yi}. \tag{4:3}$$

Now if in the integrand function we substitute this expression for g_{yi} together with the expression for $\eta^{i'}$ found in the equation

$$D\eta^i/dt = \eta^{i'} + \eta^k \Gamma^{*}{}_{k}{}^{i}{}_{h} y^{h'} + \eta^k C_k{}^i{}_h Dy^{h'}/dt$$

we find that

$$g_{yi}\eta^i + g_i\eta^{i'} = y^{m'}y^{h'}\Gamma^*_{hmi}\eta^i + g_i D\eta^i/dt$$
$$- g_i\eta^k \Gamma^{*}{}_{h}{}^{i}{}_{k} y^{h'} - g_i\eta^k C_k{}^i{}_h Dy^{h'}/dt.$$

By means of equations (1:4), (3:3) and property (H) we derive the relationships

$$g_i \eta^k \Gamma^{*\,i}_{k\ h} y^{h\prime} = y^{m\prime} y^{h\prime} \Gamma^*_{hmk} \eta^k,$$

$$g_i C_{k\ h}^{\ i} = y^{m\prime} C_{kmh} = 0. \tag{4:4}$$

Substituting from the last three equations in (4:2) and using relation (2:5) we deduce easily

THEOREM 4:1. *The integral*

$$I(a) = \int_{t_1}^{t_2} [2g(y + a\eta,\ y' + a\eta')]^{1/2} dt$$

has for its first derivative at $a = 0$ *the form*

$$I'(0) = \int_{t_1}^{t_2} \ell_i (D\eta^i/dt) dt \tag{4:5}$$

where the ℓ_i *are the covariant components of the unit vector tangent to the curve* C.

The Euler equations of Wren are [8, p. 169]

$$P_i \equiv d[g_i(2g)^{-1/2}]/dt - g_{y^i}(2g)^{-1/2} = 0 \qquad (i = 1, \ldots, n), \tag{4:6}$$

or when the parameter is so specialized that $(2g)^{1/2} = 1$

$$dg_i/dt - g_{y^i} = 0 \qquad (i = 1, \ldots, n).$$

Taking the absolute derivative of $g_i(2g)^{-1/2} = \ell_i$ with respect to t we have, from equations (3:5) and (4:4),

$$D\ell_i/dt = d\ell_i/dt - \ell_k \Gamma^{*\,k}_{i\ h} y^{h\prime}.$$

This together with equation (4:3) gives

THEOREM 4:2. *The Euler differential equations*

$$P_i \equiv d[g_i(2g)^{-1/2}]/dt - g_{y^i}(2g)^{-1/2} = 0 \qquad (i = 1, \ldots, n)$$

for the integral I, *equation* (1:3), *are identical with the equations*

(4:7) $$D\ell_i/dt = 0 \qquad (i = 1, \ldots, n)$$

and hence the P_i are the components of a covariant tensor.

There is a second form for the P_i displayed by Wren |8, p. 171|, namely,

(4:8) $$P_i = (2gg_{ij} - g_i g_j)Q^j(2g)^{-3/2}$$

where

$$Q^i = y^{i''} + \{jk,i\}\, y^{j'}y^{k'}.$$

This form, though different in appearance, is exactly the same as $D\ell_i/dt$, as we shall now show. Wren's symbol $\{jk,i\}$ has the same value as our $\gamma_j{}^i{}_k$, and when $\Gamma^*{}_j{}^i{}_k y^{j'}y^{k'}$ is found from equation (3:7) it is seen to be just $\gamma_j{}^i{}_k y^{j'}y^{k'}$. Hence equation(3:4) shows that

(4:9) $$Q^j = Dy^{j'}/dt.$$

By means of equation (2:5) and postulate (A) it is readily verified that

(4:10) $$D\ell_i/dt = (2g)^{-1/2}Dg_i/dt - g_i(2g)^{-3/2}dg/dt.$$

But from equations (1:4) $2g = g_j y^{j'}$ and hence

$$2dg/dt = g_j Dy^{j'}/dt + y^{j'}Dg_i/dt.$$

Moreover, since by (1:4) $g_i = g_{ij}y^{j'}$, it follows with the aid of equation (3:15) that

$$Dg_i/dt = g_{ij}Dy^{j'}/dt.$$

Substituting these last two results in equation (4:10) we see that

$$D\ell_i/dt = (2gg_{ij} - g_i g_j)(2g)^{-3/2} Dy^{j'}/dt$$

which by equation (4:9) is the same as (4:8).

We note here for future use the fact that equations (4:7) are equivalent to

$$D\ell^i/dt = 0 \tag{4:11}$$

since $\ell_i = g_{ij}\ell^j$ and $Dg_{ij}/dt = 0$.

5. The Weierstrass E-function and Legendre's quadratic form. The E-function and the quadratic form as given by Wren [8, p. 174] are

$$E(y,y',Y') = [2g(y,y')]^{1/2} - [g_i(y,y')Y^{i'}][2g(y,y')]^{-1/2}, \tag{5:1}$$

$$\begin{aligned} Q(y,y',\xi) = {} & g_{ij}(y,y')\xi^i \xi^j [2g(y,y')]^{-1/2} \\ & - [g_i(y,y')\xi^i]^2 [2g(y,y')]^{-3/2}. \end{aligned} \tag{5:2}$$

The Weierstrass E-function is evidently an invariant if the $Y^{i'}$ are the components of a contravariant tensor and they may be so taken since they define a direction. The ξ^i in $Q(y,y',\xi)$ are variations like the η^i and hence may also be considered as the components of a contravariant tensor. We state the result as

THEOREM 5:1. The functions $E(y,y',Y')$ and $Q(y,y',\xi)$ defined by equations (5:1) and (5:2) are invariants when the functions $Y^{i'}$ and ξ^i are the components of contravariant tensors of rank one.

6. The second variation. According to Wren [8, p. 169] the second variation has the form

$$I''(0) = \int_{t_1}^{t_2} [2\omega(2g)^{-1/2} - (g_{y^i}\eta^i + g_i\eta^{i'})^2(2g)^{-3/2}]\,dt$$

where

$$(6:1) \qquad 2\omega = g_{y^iy^j}\eta^i\eta^j + 2g_{iy^j}\eta^{i'}\eta^j + g_{ij}\eta^{i'}\eta^{j'}.$$

Along an extremal where equations (4:6) are satisfied $I''(0)$ becomes

$$(6:2) \qquad I''(0) = \int_{t_1}^{t_2}[2\omega - (\ell_i\eta^i)'^2](2g)^{-1/2}dt.$$

We now propose to reduce the expression for 2ω to one involving tensors so that, as is the case with $(\ell_i\eta^i)'$, its invariance will be at once obvious from its form. The first step in the somewhat long reduction will be that of replacing the $\eta^{i'}$ where they occur in 2ω by expressions obtained for them by means of the equations

$$D\eta^i/dt = \eta^{i'} + \eta^k\Gamma^{*\,i}_{k\,h}y^{h'} + \eta^kC_k{}^i{}_hDy^{h'}/dt.$$

Now along an extremal arc E_{12} where $D\ell^h/dt = 0$, the last term drops out since then

$$D\ell^h/dt = D[y^{h'}(2g)^{-1/2}]/dt = (Dy^{h'}/dt)(2g)^{-1/2} + y^{h'}d(2g)^{-1/2}/dt = 0$$

in which we have used the fact that the absolute derivative of an invariant is equal to the ordinary derivative. Hence by (3:3) and the homogeneity property (1:4) it follows from the last equation that along an extremal $C_k{}^i{}_hDy^{h'}/dt = 0$ and we may write

$$(6:3) \qquad D\eta^i/dt = \eta^{i'} + \eta^k\Gamma^{*\,i}_{k\,h}y^{h'}.$$

Substituting the value of $\eta^{i'}$ from this equation in (6:1) we obtain

$$\begin{aligned}2\omega = {} & \eta^i\eta^j[g_{y^iy^j} - 2g_{ky^j}\Gamma^{*\,k}_{i\,h}y^{h'} + g_{kr}\Gamma^{*\,r}_{i\,h}\Gamma^{*\,k}_{j\,m}y^{h'}y^{m'}]\\ (6:4) \qquad & + 2(D\eta^i/dt)\eta^j(g_{iy^j} - g_{ir}\Gamma^{*\,r}_{j\,h}y^{h'})\\ & + g_{ij}(D\eta^i/dt)(D\eta^j/dt),\end{aligned}$$

where the last term is an invariant and will be retained in its present form in the final result.

Our next step is to replace the term in $(D\eta^i/dt)\eta^j$ by a derivative of a term with respect to t plus terms in $\eta^i\eta^j$. An expression for g_{iy^j} in terms of the functions Γ^* can be deduced from (3:2) multiplied by $y^{i'}$ and summed, (3:5), (3:3), and (1:4). The value so obtained is

$$g_{iy^j} = y^{m'}(\Gamma^*_{mij} + \Gamma^*_{imj}). \tag{6:5}$$

It follows from this, because by postulate (H) $\Gamma^*_{mij} = \Gamma^*_{jim}$, and from (6:3) that the second row in the expression (6:4) divided by $(2g)^{1/2}$ has the value

$$\begin{aligned} &2(D\eta^i/dt)\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2} \\ &= 2\eta^{i'}\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2} + 2\eta^k\eta^j y^{m'}\Gamma^*_{imj}\Gamma^{*\,i}_{k\,h}y^{h'}(2g)^{-1/2} \\ &= d[\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt - \eta^i\eta^j d[y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt \\ &\qquad + 2\eta^k\eta^j y^{m'}\Gamma^*_{imj}\Gamma^{*\,i}_{k\,h}y^{h'}(2g)^{-1/2}. \end{aligned}$$

If we insert this expression for the second row in (6:4) and modify the notation for the quadratic form in $\eta^i\eta^j$ so that its coefficients are symmetric, then (6:4) divided by $(2g)^{1/2}$ takes the form indicated in the following theorem:

THEOREM 6:1. _The quadratic form_ $2\omega(2g)^{-1/2}$ _in the integrand function of the second variation may be written in the form_

$$\begin{aligned} 2\omega(2g)^{-1/2} &= [g_{ij}(D\eta^i/dt)(D\eta^j/dt) - \eta^i\eta^j R_{ij}](2g)^{-1/2} \\ &\qquad + d[\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt \end{aligned} \tag{6:6}$$

where

$$
\begin{aligned}
R_{ij} = {} & g_{y^iy^j} - g_{ky^j}\Gamma^{*}{}_{i}{}^{k}{}_{h}y^{h'} - g_{ky^i}\Gamma^{*}{}_{j}{}^{k}{}_{h}y^{h'} \\
(6:7) \qquad & + y^{m'}\Gamma^{*}{}_{jkm}\Gamma^{*}{}_{i}{}^{k}{}_{h}y^{h'} + y^{m'}\Gamma^{*}{}_{imk}\Gamma^{*}{}_{j}{}^{k}{}_{h}y^{h'} \\
& + y^{m'}\Gamma^{*}{}_{jmk}\Gamma^{*}{}_{i}{}^{k}{}_{h}y^{h'} - (2g)^{1/2}d[y^{m'}\Gamma^{*}{}_{imj}(2g)^{-1/2}]/dt
\end{aligned}
$$

is a tensor.

The proof has already been made except for the statement that R_{ij} is a tensor. In order to prove this we show first that the last term in (6:6) is an invariant. Since $y^i(t,a) = y^i(t) + a\eta^i(t)$ we see that

$$
\begin{aligned}
g_k(D\eta^k/da) &= g_{km}y^{m'}[\partial\eta^k/\partial a + \eta^i\Gamma^{*}{}_{i}{}^{k}{}_{j}(\partial y^j/\partial a) + \eta^iC_i{}^k{}_j(\partial y^{j'}/\partial a)] \\
&= \eta^iy^{m'}\Gamma^{*}{}_{imj}\eta^j,
\end{aligned}
$$

the last value being due to the fact that $\partial\eta^k/\partial a = 0$ and $y^{m'}C_{imj} = 0$. Hence the last term in (6:6) is an invariant, and as a result $\eta^i\eta^jR_{ij}$ is also. To prove that R_{ij} is a transformation (2:1) and find

$$
\begin{aligned}
\bar{\eta}^i\bar{\eta}^j\bar{R}_{ij} &= \eta^\alpha\eta^\beta R_{\alpha\beta} \\
&= \bar{\eta}^i\bar{\eta}^j(\partial y^\alpha/\partial\bar{y}^i)(\partial y^\beta/\partial\bar{y}^j)R_{\alpha\beta}\,.
\end{aligned}
$$

This is an identity for arbitrary elements $\bar{\eta}^i$ and hence, because of the symmetry of R_{ij},

$$\bar{R}_{ij} = R_{\alpha\beta}(\partial y^\alpha/\partial\bar{y}^i)(\partial y^\beta/\partial\bar{y}^j)$$

which shows that R_{ij} is a covariant tensor of rank two. This concludes the proof of the theorem.

As a corollary to Theorem (6:1) we have

THEOREM 6:2. The second variation

$$I''(0) = \int_{t_1}^{t_2}[2\omega - (\ell_i\eta^i)'^2](2g)^{-1/2}dt$$

may be written in the form

$$\begin{aligned}(6{:}8)\qquad I''(0) = \int_{t_1}^{t_2} [&g_{ij}(D\eta^i/dt)(D\eta^j/dt) \\ &- \eta^i\eta^j R_{ij} - (\ell_i\eta^i)'^2](2g)^{-1/2}dt.\end{aligned}$$

This follows at once from (6:2) and (6:6) because

$$\int_{t_1}^{t_2} [d(\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2})/dt]dt = \eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}\Big|_{t_1}^{t_2} = 0$$

since $\eta^i(t_1) = \eta^i(t_2) = 0$.

It is interesting to note that the tensor R_{ij} is closely related to the tensor R_{himj} of Cartan [9, p. 36, XIX]. In order to show this we shall express the elements of R_{ij} in (6:7) in terms of the functions Γ^* and their derivatives so that the relationship will be apparent. Let us first consider the derivative

$$\begin{aligned}&d[y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt \\ (6{:}9)\qquad &= \Gamma^*_{imj}d[y^{m'}(2g)^{-1/2}]/dt + y^{m'}(2g)^{-1/2}[(\partial\Gamma^*_{imj}/\partial y^r)y^{r'} \\ &\qquad + (\partial\Gamma^*_{imj}/\partial y^{r'})y^{r''}].\end{aligned}$$

Now along the extremal arc E_{12} we have $D\ell^m/dt = 0$. Therefore from (2:4), (3:5), and (1:4) it follows that

$$d[y^{m'}(2g)^{-1/2}]/dt = -y^{k'}\Gamma^{*}{}_{k}{}^{m}{}_{h}y^{h'}(2g)^{-1/2},$$

and after expanding the left hand member

$$y^{m''}(2g)^{-1/2} = -y^{k'}\Gamma^{*}{}_{k}{}^{m}{}_{h}y^{h'}(2g)^{-1/2} - y^{m'}d(2g)^{-1/2}/dt.$$

When we substitute from these last two equations the equation (6:9) becomes

$$\begin{aligned} & d[y^{m'}\Gamma^{*}{}_{imj}(2g)^{-1/2}]/dt \\ (6:10)\qquad & = (2g)^{-1/2}[-y^{m'}y^{h'}\Gamma^{*}{}_{m}{}^{k}{}_{h}\Gamma^{*}{}_{ikh} + y^{m'}y^{h'}(\partial\Gamma^{*}{}_{ihj}/\partial y^{m}) \\ & \qquad - y^{m'}y^{h'}(\partial\Gamma^{*}{}_{ihj}/\partial y^{r'})(\partial G^{r}/\partial y^{m'})] \end{aligned}$$

if we make use of (3:8) and the homogeneity of order zero of $\Gamma^{*}{}_{imj}$ which follows from (3:7).

Employing (6:10), (6:5), and the result of differentiating (4:3) with respect to y^{j} we are now able to rewrite the expression for R_{ij} in (6:7) as

$$\begin{aligned} & y^{m'}y^{h'}[\partial\Gamma^{*}{}_{hmi}/\partial y^{j} - \partial\Gamma^{*}{}_{ihj}/\partial y^{m} + (\partial\Gamma^{*}{}_{ihj}/\partial y^{r'})(\partial G^{r}/\partial y^{m'}) \\ (6:11)\qquad & \qquad -\Gamma^{*}{}_{mkj}\Gamma^{*}{}_{i}{}^{k}{}_{h} + \Gamma^{*}{}_{ikj}\Gamma^{*}{}_{m}{}^{k}{}_{h}]. \end{aligned}$$

Again applying (6:5) we see that

$$\partial\,[y^{m'}(\Gamma^{*}{}_{imh} + \Gamma^{*}{}_{mih})]/\partial y^{j} = g_{iy^{j}y^{h}}$$

and therefore the first term in (6:11) has the value

$$\begin{aligned} & y^{m'}y^{h'}(\partial\Gamma^{*}{}_{hmi}/\partial y^{j}) = y^{m'}y^{h'}(\partial\Gamma^{*}{}_{imh}/\partial y^{j}) \\ (6:12)\qquad & \qquad = -y^{m'}y^{h'}(\partial\Gamma^{*}{}_{mih}/\partial y^{j}) + g_{iy^{j}y^{h}}y^{h'}. \end{aligned}$$

Similarly the second term in (6:11) is

$$(6:13)\quad y^{m'}y^{h'}(\partial\Gamma^{*}{}_{ihj}/\partial y^{m}) = -\,y^{m'}y^{h'}(\partial\Gamma^{*}{}_{hij}/\partial y^{m}) + g_{iy^{j}y^{m}}y^{m'}.$$

To modify the third term in (6:11) we differentiate (6:5) with respect to $y^{r'}$ and apply (3:8), then (3:6) and (3:2) so that we have

$$(6:14)\quad y^{h'}(\partial\Gamma^{*}{}_{ihj}/\partial y^{r'}) = 2C_{ikr}(\partial G^{k}/\partial y^{j'}) - y^{h'}(\partial\Gamma^{*}{}_{hij}/\partial y^{r'}).$$

Furthermore it can be verified by a differentiation with respect to $y^{r'}$ of the Γ^*_{mih} as found in equation (3:7) that

$$\begin{aligned}\partial\Gamma^*_{him}/\partial y^{r'} = {} & (1/2)(g_{hiry^m} + g_{imry^h} - g_{hmry^i}) + C_{hmrk}(\partial G^k/\partial y^{i'}) \\ & - C_{hirk}(\partial G^k/\partial y^{m'}) - C_{imrk}(\partial G^k/\partial y^{h'}) \\ & + C_{hmk}(\partial^2 G^k/\partial y^{i'}\partial y^{r'}) - C_{hik}(\partial^2 G^k/\partial y^{m'}\partial y^{r'}) \\ & - C_{imk}(\partial^2 G^k/\partial y^{h'}\partial y^{r'}).\end{aligned}$$

Now when we multiply by $y^{m'}y^{h'}$ and add and use the homogeneity of order minus one of C_{hik}, this last equation gives us

$$(6{:}15) \qquad y^{m'}y^{h'}(\partial\Gamma^*_{mih}/\partial y^{r'}) = 2C_{irk}(\partial G^k/\partial y^{m'})y^{m'}.$$

Finally by means of equations (6:12) to (6:15) we obtain in place of (6:11)

$$\begin{aligned}R_{ij} = -y^{h'}y^{m'}\, [& \partial\Gamma^*_{him}/\partial y^j - (\partial\Gamma^*_{mih}/\partial y^{r'})(\partial G^r/\partial y^{j'}) \\ & - \partial\Gamma^*_{hij}/\partial y^m + (\partial\Gamma^*_{hij}/\partial y^{r'})(\partial G^r/\partial y^{m'}) \\ & + \Gamma^*_{hkj}\Gamma^{*\,k}_{i\ m} - \Gamma^*_{ikj}\Gamma^{*\,k}_{m\ h}].\end{aligned}$$

A comparison of this expression with Cartan's equation XIX [9, p. 36] and a use of equations (3:3) and (1:4) give us the following theorem:

THEOREM 6:3. The tensor R_{ij} of equation (6:7) when evaluated along an extremal arc E_{12} is exactly

$$y^{h'}y^{m'}R_{himj} = -(2g)R_{0i0j}$$

where R_{himj} is Cartan's tensor given by the formula

$$\begin{aligned}R_{himj} = {} & \partial\Gamma^{*}_{him}/\partial y^{j} - (\partial\Gamma^{*}_{him}/\partial y^{r\prime})(\partial G^{r}/\partial y^{j\prime}) - \partial\Gamma^{*}_{hij}/\partial y^{m} \\ & + (\partial\Gamma^{*}_{hij}/\partial y^{r\prime})(\partial G^{r}/\partial y^{m\prime}) + \Gamma^{*}_{hkj}\Gamma^{*}{}_{i}{}^{k}{}_{m} - \Gamma^{*}_{ikj}\Gamma^{*}{}_{h}{}^{k}{}_{m} \\ & + C_{hir}[\partial^{2}G^{r}/\partial y^{m\prime}\partial y^{j} - \partial^{2}G^{r}/\partial y^{j\prime}\partial y^{m} - (\partial^{2}G^{r}/\partial y^{m}\partial y^{k})(\partial G^{k}/\partial y^{j\prime}) \\ & + (\partial^{2}G^{r}/\partial y^{j\prime}\partial y^{k\prime})(\partial G^{k}/\partial y^{m\prime})]\end{aligned}$$

and R_{oioj} *is* $\ell^{h}\ell^{m}R_{himj}$.

7. *The Jacobi differential equations*. In considering the Jacobi differential equations we suppose, as does Wren [8, p. 175], that the parameter t is so specialized that $(2g)^{1/2} = 1$. The second variation of equation (6:8) then takes the form

$$(7{:}1)\quad I''(0) = \int_{t_1}^{t_2} [g_{ij}(D\eta^{i}/dt)(D\eta^{j}/dt) - \eta^{i}\eta^{j}R_{ij} - (g_{i}\eta^{i})'^{2}]dt,$$

since from equation (2:5) ℓ_{i} is now the same as g_{i}. Introducing the notation

$$2\Omega = g_{ij}(D\eta^{i}/dt)(D\eta^{j}/dt) - \eta^{i}\eta^{j}R_{ij},$$

we see from equation (6:6) that

$$(7{:}2)\qquad 2\omega = 2\Omega + d(\eta^{i}\eta^{j}y^{m\prime}\Gamma^{*}_{imj})/dt.$$

Since the Euler differential equations are identically zero when the integrand function is a derivative, it follows that the Euler differential equations for $I''(0)$ in the form (7:1) are exactly the same as the equations

$$J_{i}(\eta) = d\,\omega_{\eta^{i\prime}}/dt - \omega_{\eta^{i}} - (g_{j}\eta^{j})''g_{i} = 0$$

given by Wren [8, p. 176].

With the aid of equation (7:2) and the statement at the end of the last paragraph it is clear that

$$(7{:}3) \qquad J_i(\eta) = d\Omega_{\eta i'}/dt - \Omega_{\eta i} - g_i(g_j\eta^j)'' = 0.$$

It can be easily verified by means of the definition of Ω and the equation (6:3) that

$$(7{:}4) \qquad \begin{aligned} \Omega_{\eta i'} &= g_{ij}(D\eta^j/dt), \\ \Omega_{\eta i} &= y^{h'}\Gamma^*_{ijh}(D\eta^j/dt) - R_{ij}\eta^j. \end{aligned}$$

Obviously $\Omega_{\eta i'}$ is a covariant tensor of rank one and we may take its covariant derivative. Using (3:5) and the fact that along an extremal with the parameter so chosen that $(2g)^{1/2} = 1$

$$D\ell^i/dt = Dy^{i'}/dt = 0,$$

we may write

$$D\Omega_{\eta i'}/dt = d\Omega_{\eta i'}/dt - \Omega_{\eta k'}\Gamma^{*}{}_{i}{}^{k}{}_{h}y^{h'}.$$

By means of postulate (D) and equations (3:15) and (7:4) it follows from the last equation that

$$(7{:}5) \qquad d\Omega_{\eta i'}/dt = g_{ij}D^2\eta^j/dt^2 + \Gamma^*_{ijh}y^{h'}D\eta^j/dt.$$

Substituting from (7:5) and (7:4) in (7:3) we find the following result:

THEOREM 7:1. The first members of the Jacobi equations of the problem of the calculus of variations in the form proposed by Wren are the elements of the tensor

$$(7{:}6) \qquad J_i(\eta) = g_{ij}D^2\eta^j/dt^2 + R_{ij}\eta^j - g_i(g_j\eta^j)''.$$

These equations satisfy along an extremal are the relation

$y^{i\prime}J_i(\eta) = 0$ *and are not independent, as Wren has shown* [8,p. 179]. *Wren has also shown* [8, pp. 176-7] *that all the solutions of the equations* $J_i(\eta) = 0$ *satisfying the orthogonality condition* $g_i\eta^i = 0$ *are solutions of the equations of Wren* [8, p. 177]

$$K_i(\eta) = d\omega_{\eta i'}/dt - \omega_{\eta i} = 0$$

with initial conditions $g_i\eta^i = (g_i\eta^i)' = 0$ *at* t_1. *The functions* $K_i(\eta)$ *are the elements of the tensor*

$$K_i(\eta) = g_{ij}D^2\eta^j/dt^2 + R_{ij}\eta^j.$$

8. *The determinant* $\Delta(t,t_1)$. The determinant $\Delta(t,t_1)$ whose zeros determine the points 3 conjugate to the initial point 1 on the extremal are E_{13} has in Wren's notation [8, p. 184] the form

$$(t,t_1) = \begin{vmatrix} u^{ik}(t) & v^{ik}(t) & y^{i\prime}(t) & 0 \\ u^{ik}(t_1) & v^{ik}(t_1) & 0 & y^{i\prime}(t_1) \end{vmatrix}$$

in which the matrices (u^{ik}), (v^{ik}) consist of $n-1$ columns of solutions of the equations $K_i(\eta) = 0$. As Wren notes [8, p. 181] the columns may be made into orthogonal solutions [8, p. 174] of $J_i(\eta) = 0$ by subtracting from each suitable multiples of the last two columns. These orthogonal solutions of the Jacobi equations are functions which will be denoted by

$$\eta^i_r(t) \qquad (i = 1,\ldots,n;\ r = 1,\ldots,2n-2).$$

Since, as equation (7:6) shows, the left-hand members of the Jacobi equations are the components of a covariant tensor, it follows that if η^i_r, r fixed, is a solution of $J_i(\eta) = 0$, its transform $\bar{\eta}^k_r = \eta^i_r\,\partial\bar{y}^k/\partial y^i$ is a solution of the transformed equation $J_k(\eta) = 0$. We rewrite $\Delta(t,t_1)$ in the form

(8:1)
$$\begin{vmatrix} \eta^i_r(t) & y^{i'}(t) & 0 \\ \eta^i_r(t_1) & 0 & y^{i'}(t_1) \end{vmatrix}$$

If a transformation of variables (2:1) is made with $\bar\eta^k_r = \eta^i_r \partial\bar y^k/\partial y^i$ and $\bar y^{k'} = y^{i'}\partial\bar y^k/\partial y^i$, the transform $\bar\Delta(t,t_1)$ of the determinant (8:1) becomes

$$\begin{vmatrix} \eta^i_r\partial\bar y^k/\partial y^i & y^{i'}\partial\bar y^k/\partial y^i & 0 \\ (\eta^i_r\partial\bar y^k/\partial y^i)_1 & 0 & (\bar y^{i'}\partial\bar y^k/\partial y^i)_1 \end{vmatrix}$$

where the parentheses with a subscript 1 indicate that the quantity inside is to be evaluated at $t = t_1$. It can easily be seen that $\bar\Delta(t,t_1)$ is the product of the determinant

(8:2)
$$\begin{vmatrix} \partial\bar y^k/\partial y^i & 0 \\ 0 & (\partial\bar y^k/\partial y^i)_1 \end{vmatrix}$$

by the determinant (8:1). When the determinant (8:2) is evaluated by the Laplace expansion we have the following result:

THEOREM 7:1. *The determinant* $\Delta(t,t_1)$ *whose zeros determine the points* 3 *conjugate to the initial point* 1 *on the extremal arc* E_{12} *transforms by the law*

$$\bar\Delta(t,t_1) = J(\bar y,y)[J(\bar y,y)]_1\Delta(t,t_1)$$

where $J(\bar y,y)$ *is the Jacobian of the transformation* $\bar y = \bar y(y)$. *Consequently the transforms of the conjugate points for the problem in terms of the original variables* y^i *are conjugate points for the problem in the transformed variables* $\bar y^i$.

9. *The canonical differential equations*. For the discussion of the Hamilton-Jacobi theory Wren [8, p. 186] replaces the original variables y^i, $y^{i\prime}$ by canonical variables y^i, z_i in which the z_i are defined for the region R by the equations

$$(8:1) \qquad z_i = g_i(y,y').$$

Since $|g_{ij}| \neq 0$ these equations are solvable, at least locally, for the $y^{i\prime}$ as functions of y^i and z_i so that

$$(8:2) \qquad y^{i\prime} = P^i(y,z).$$

The Hamilton function may be defined to be

$$(8:3) \qquad H(y,z) = [2g(y,P)]^{1/2}.$$

This differs unessentially from Wren's definition. His function is this one minus unity. According to his formula [8, p. 187]

$$(8:4) \qquad H_{y^i} = - g_{y^i}(y,P)/H, \qquad H_{z_i} = P^i(y,z)/H.$$

The canonical differential equations are then [8, p. 188]

$$(8:5) \qquad y^{i\prime} - H_{z_i}(y,z) = 0, \quad z_i' + H_{y^i}(y,z) = 0, \quad H(y,z) = 1.$$

It is evident from equations (8:1) and (8:2) that z_i and P^i are the components, respectively, of a covariant and a contravariant tensor of rank one. Hence from (8:4) the H_{z_i} are the components of a contravariant unit tensor. As a result the left-hand members of the first set of equations (8:5) are the components of a contravariant tensor of rank one. A similar statement cannot be made immediately for the second set, since neither the $z_i'(t)$ nor the H_{y^i} are themselves the components of a tensor. However, their sum is a covariant tensor, as will now be shown.

In order to see this we form the absolute derivative Dz_i/dt, the fundamental function being $g = g(y,P)$. We have from equation (3:5)

$$(8{:}6) \qquad Dz_i/dt = dz_i/dt - z_k\Gamma^{*}{}_i{}^k{}_h\, dy^h/dt - z_k C_i{}^k{}_h\, DP^h/dt.$$

But since $z_k = g_k(y,P)$ it follows from (4:4) that $z_k C_i{}^k{}_h = 0$. Moreover, with P^i in place of $y^{i'}$ in the function g, we see from (1:4) that $z_k = P^j g_{kj}$, and hence equation (8:6) becomes

$$(8{:}7) \qquad Dz_i/dt = dz_i/dt - P^j\Gamma^{*}{}_{ijh}P^h.$$

When equations (3:7) are multiplied by P^j and P^h and summed it follows from the homogeneity properties (1:4) that

$$(8{:}8) \qquad Dz_i/dt = dz_i/dt - g_{y^i}(y,P).$$

Hence we have the result:

THEOREM 8:1. *The left-hand members of the* $2n+1$ *canonical differential equations* (8:5) *of Wren may be expressed in the form*

$$y^{i'} - H_{z_i}(y,z), \qquad Dz_i/dt, \qquad H(y,z) \equiv [2g(y,P)]^{1/2}.$$

The first two sets of n *functions are the components, respectively, of a contravariant and a covariant tensor. The last function,* $H(y,z)$, *is an invariant.*

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INVARIANT MULTIPLE INTEGRALS
IN THE
CALCULUS OF VARIATIONS

BY

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INVARIANT MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

Introduction. The calculus of variations problem with which we are concerned is that of minimizing a multiple integral of the form

$$\int_A f(t_1, \dots, t_m; x^1, \dots, x^n; x^1_1, \dots x^n_m)dt_1 \dots dt_m$$

in a class of admissible n-dimensional spaces $x = x(t)$, for each of which the integral has a well defined value, and all of which pass through a fixed $(n-1)$-dimensional bounding space L in tx-space. In this integral x^i denotes the unknown function $x^i(t)$ and x^i_a stands for the partial derivative of x^i with respect to t_a, while A is the region of integration in t-space.

The theory of fields and their associated invariant integrals for multiple integral problems have been studied from different points of view by Bliss (I)[1], Caratheodory (II) and Weyl (III). In the paper by Bliss referred to here, he restricts himself to double integral problems but, more recently, he has given a general definition of a field for the problem here studied. Dr. M. F. Smiley, while studying under Professor Bliss, obtained a set of conditions which every integrand of an invariant integral must satisfy. In this paper we show, by a method analogous to that used by Bliss (I), that these same conditions on the integrand are sufficient to make an integral invariant. We then make a study of fields having associated with them invariant integrals of this general type. Solutions of certain systems of partial differential equations which arise in this study are found by a method suggested by Hestenes.

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[1] Roman numerals in parenthesis refer to the bibliography at the end of the paper.

In the first section of this paper we formulate the problem in detail, express the Euler and Weierstrass necessary conditions, define a field and show what would be needed to make a sufficiency proof for minimizing the above integral. In the second section we give necessary and sufficient conditions for the value of an integral of the above type to depend only on the bounding space of the class of admissible spaces under consideration.

It is well known that when $n = m = 1$ every family of extremals which simply covers a region F in tx-space defines a unique field over F. When $n > m = 1$ a family of extremals may simply cover a region F without defining a field. However, if such a family defines a field over F, it is unique. In Sections 3 and 4 below, it is shown that when $m > 1$ every family of extremals, with suitable analytic properties, which simply covers a region F defines a field over F. This field is unique only in case $n = 1$. The form of every invariant integral determined by a family of extremals of this type is also given. When $m > 1$, $n > 1$ the most general invariant integral associated with a field is expressed in terms of solutions of systems of linear partial differential equations of the first order. Solutions of these equations are given in the final section. The above results are first obtained by specializing the coordinates in the manner described in Section 3. In Section 4, they are interpreted in terms of the original coordinates. As a by-product, we obtain the usual necessary and sufficient conditions for the formation of a field when $n > m = 1$.

1. Formulation of the problem. Let us consider the m-tuple integral

$$I = \int_T f(t,x,x_t)dt \tag{1:1}$$

where

$$t \equiv (t_1, \ldots, t_m),\ x \equiv (x^1, \ldots, x^n),\ x_t \equiv (x^1_1, \ldots, x^n_m),$$

$$dt \equiv dt_1 \ldots dt_m,\ x^i_a \equiv \partial x^i/\partial t_a \quad (a = 1, \ldots, m;\ i = 1, \ldots, n)$$

and T is an m-dimensional region in t-space. The integrand function f is assumed to have continuous partial derivatives, of as many orders as is needed for the arguments of the following pages, in a region R of an $(m+n+mn)$-dimensional space of elements (t,x,x_t).

An m-space defined by functions of the form

$$x(t) \qquad (t \text{ in } T) \tag{1:2}$$

will be said to be admissible if the functions $x(t)$ have continuous partial derivatives of the second order in a bounded closed connected region T of t-space and if further all the elements (t,x,x_t) belonging to the space are interior to R.

Let A denote the $(m-1)$-space in t-space which bounds T and let L represent a fixed $(m-1)$-space in tx-space whose projection on t-space is A. The points of A and L are to be in one-to-one correspondence. The calculus of variations problem is then to find in the class of admissible m-spaces (1:2) which pass through a given $(m-1)$-space L, one which minimizes the integral (1:1). It is understood that in the integrand of the integral I the variable x^i is replaced by $x^i(t)$ from (1:2) and x^i_a is replaced by $dx^i(t)/dt_a$. Here and in the following pages the index a with or without a subscript has the range of integers from 1 to m, and the index i similarly the range from 1 to n.

The Euler necessary condition for the above problem may be stated as follows (VII, p. 17):

THEOREM 1:1. *On a minimizing m-space* $x = x(t)$, *(t in T), we must have*

$$df_{x^i_a}/dt_a = f_{x^i}. \tag{1:3}$$

As in tensor analysis a repeated index a or i, with or without a subscript, indicates summation over the specified range. An admissible m-space which satisfies the Euler equations (1:3) is called an *extremal*.

For the above problem the E-function of Weierstrass is expressed by the equation

$$E(t,x,x_t,X_t) = f(t,x,X_t) - f(t,x,x_t) - (X^i_a - x^i_a)\, f_{x^i_a}(t,x,x_t).$$

The Weierstrass necessary condition may be stated as follows:[1]

THEOREM 1:2. *At every element* (t,x,x_t) *of a minimizing m-space* $x = x(t)$, *(t in T), the condition*

$$E(t,x,x_t,X_t) \geqq 0$$

is satisfied for every X_t *such that the matrix* $\|X^i_a - x^i_a\|$ *is of rank* 1.

Bliss has given the following definition of a field. *A field for the integral I is a region F of tx-space with which there is associated a set of slope functions* $p^i_a(t,x)$ *having continuous partial derivatives of the second order in F and an m-tuple integral*

$$I^* = \int_T D(t,x,x_t)\,dt \tag{1:4}$$

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[1] McShane (VI, p. 584); for n = 2, Coral (IV, p. 589).

such that

$$D(t,x,p) = f(t,x,p), \qquad D_{x^i_a}(t,x,p) = f_{x^i_a}(t,x,p) \tag{1:5}$$

and further such that the value of I^* *is the same for all admissible m-spaces* $x = x(t)$ *in* F *with a common boundary* L.

The differential equations of the field are

$$x^i_a = p^i_a(t,x) \qquad (t \text{ in } T). \tag{1:6}$$

Every solution of these equations is an extremal, for, if we differentiate the first of equations (1:5) with respect to x^i and use the other equations of (1:5) we find

$$D_{x^i}(t,x,p) = f_{x^i}(t,x,p)$$

and hence

$$(d/dt_a)f_{x^i_a}(t,x,p) - f_{x^i}(t,x,p) = (d/dt_a)D_{x^i_a}(t,x,p) - D_{x^i}(t,x,p) = 0.$$

A solution of equations (1:6) is called an *extremal of the field*. We see by means of the first equation of (1:5) that on an extremal space of the field the values of I and I^* are equal.

If we denote by I(E) and I(S), respectively, the values of I computed for an extremal space E of the field with boundary L and an admissible m-space S of the form (1:2) with the same boundary, then

$$I(S) - I(E) = I(S) - I^*(E) = I(S) - I^*(S) = \int_T (f - D)\,dt$$

where (t,x,x_t) in the integrand function are those belonging to S. It is evident that if there exists an invariant integral of the form (1:4) such that $f - D \geqq 0$, then E minimizes the integral I.

2. *Invariant integrals.* Let R_0 represent a region of admissible elements (t,x,x_t) characterized by conditions of the form

$$t \text{ in } T; \quad X^{1i}(t) < x^i < X^{2i}(t); \quad x_t \text{ arbitrary},$$

where $X^1(t)$ and $X^2(t)$ are admissible m-spaces. We shall give a proof of the following theorem:

THEOREM 2:1. *An integral* I *of the form* (1:1) *has the same value for all* m-*spaces lying in a region* R_0 *of the type described above and having the same fixed* (m - 1)-*space* L *as boundary if and only if the integrand* $f(t,x,x_t)$ *of* I *is expressible in* R_0 *in the form*

$$(2:1) \qquad f(t,x,x_t) = B(t,x) + \sum_{r=1}^{s} \frac{1}{r!} x^{i_1}_{a_1} \cdots x^{i_r}_{a_r} B^{a_1 \cdots a_r}_{i_1 \cdots i_r}(t,x)$$

where s *is the smaller of* m *and* n, *and where*

$$(2:2) \qquad (\partial/\partial t_a) B^{a a_1 \cdots a_r}_{i i_1 \cdots i_r} = B^{a_1 \cdots a_r}_{i_1 \cdots i_r i} \qquad (r = 0, 1, \ldots, s)$$

in which

$$(2:3) \qquad \begin{aligned} B^{a_1 \cdots a_r}_{i_1 \cdots i_r i} \equiv {} & (\partial/\partial x^i) B^{a_1 \cdots a_r}_{i_1 \cdots i_r} - (\partial/\partial x^{i_1}) B^{a_1 a_2 \cdots a_r}_{i \, i_2 \cdots i_r} \\ & - \cdots - (\partial/\partial x^{i_r}) B^{a_1 \cdots a_{r-1} a_r}_{i_1 \cdots i_{r-1} i} \end{aligned}$$

and the interchange of two a's or two i's simply changes the sign of the B's. *It is understood here that* $B^{a_1 \cdots a_r}_{i_1 \cdots i_r}$ *reduces to* B *for* $r = 0$.

A proof that the conditions of this theorem are necessary was made by Dr. M. F. Smiley while a student in a class conducted by Professor Bliss. For the sake of completeness this proof is given below. It will be proved, by a method analogous to that used by Bliss for the double integral problem (I, pp. 6-8), that these same conditions are sufficient.

In order to prove the necessity of the conditions stated in the theorem, we first note that for every prescribed set $(t,x,x_t,x_{tt})_0$ with $(t,x,x_t)_0$ in R_0 and t_0 interior to T there is an admissible m-space $x = x(t)$ passing through the prescribed set. Let L be an $(m-1)$-space, of the type described in the preceding section, on this admissible m-space and having the point $(t,x)_0$ in the interior of the region bounded on the space $x(t)$ by the set L. Since the integral I has the same value on all m-spaces through the $(m-1)$-space L, it follows that all such m-spaces are minimizing spaces. In particular the m-space $x = x(t)$ constructed above has this property and the Euler equations (1:3) must be satisfied on it. These equations may be written in the form

$$(2:4) \qquad f_{x^i_a t_a} + x^j_a f_{x^i_a x^j} + x^j_{ba} f_{x^i_a x^j_b} = f_{x^i}.$$

Here and in the following pages it will be convenient, at times, to use some or all of the letters b and c in place of a's with subscripts and j, k and h in place of i's with subscripts. Since the set $(t,x,x_t,x_{tt})_0$ can be selected arbitrarily it follows that equations (2:4) hold at every set (t,x,x_t,x_{tt}) with (t,x,x_t) admissible and (t,x) in R_0. Hence

$$(2:5) \qquad f_{x^i_a x^j_b} + f_{x^i_b x^j_a} = 0$$

for (t,x,x_t) admissible and (t,x) in R_0. It follows from equations (2:5) that $f_{x^{i_1}_{a_1}\cdots x^{i_r}_{a_r}}$ vanishes whenever indices a or i are repeated, and consequently that f is of degree at most s in x^i_a, where s is the smaller of m and n. Thus for (t,x,x_t) admissible and (t,x) in R_0, the integrand function f has the form given by equation (2:1) where the B's in this equation are functions of t and x with continuous partial derivatives of as many orders as are possessed by

f in R_0. It is evident from the form of equation (2:1) that for (t,x) in R_0,

$$f_{x^{i_1}_{a_1}\dots x^{i_r}_{a_r}}(t,x,0) = B^{a_1\dots a_r}_{i_1\dots i_r}(t,x). \tag{2:6}$$

It follows from equations (2:5) and (2:6) that the interchange of two a's or two i's simply changes the sign of $B^{a_1\dots a_r}_{i_1\dots i_r}$ with $r > 1$ and hence further that these same B's vanish identically for two a's or two i's equal. In particular, all of these B's in which $r > s$ vanish identically. Examination of the identities (2:3) shows that these same properties apply to $B^{a_1\dots a_r}_{i_1\dots i_r i}$. It follows that whenever $m \geqq n$ the equations (2:2) with $r = s$ are always satisfied, since at least two of the i's in the set $(i, i_1, \dots, i_s)$ are equal.

From equation (2:1) we find that

$$f_{x^i_a} = B^a_i + \sum_{r=2}^{s} \frac{1}{(r-1)!} x^{i_2}_{a_2}\dots x^{i_r}_{a_r} B^{aa_2\dots a_r}_{ii_2\dots i_r}.$$

By means of these equations and the relation (2:1), equations (2:4) become

$$\begin{gathered}(\partial/\partial t_a)B^a_i + \sum_{r=2}^{s} \frac{1}{(r-1)!} x^{i_2}_{a_2}\dots x^{i_r}_{a_r} (\partial/\partial t_a)B^{aa_2\dots a_r}_{ii_2\dots i_r} \\ + x^{i_1}_a(\partial/\partial x^{i_1})B^a_i + x^{i_1}_a \sum_{r=2}^{s} \frac{1}{(r-1)!} x^{i_2}_{a_2}\dots x^{i_r}_{a_r}(\partial/\partial x^{i_1})B^{aa_2\dots a_r}_{ii_2\dots i_r} \\ - (\partial/\partial x^i)B - \sum_{r=1}^{s} \frac{1}{r!} x^{i_1}_{a_1}\dots x^{i_r}_{a_r}(\partial/\partial x^i)B^{a_1\dots a_r}_{i_1\dots i_r} = 0.\end{gathered}$$

In the first summation, we replace r by $r+1$ and then rename the indices a_{r+1} and i_{r+1}, respectively, a_1 and i_1. In the second summation, we may bring in the preceding term by including $r = 1$, rename a as a_1 and change the form as indicated below so that the factor $1/r!$ will appear. Using the functions defined by relations

(2:3) we now have

$$(\partial/\partial t_a)B^a_i - (\partial/\partial x^i)B + \sum_{r=1}^{s} \frac{1}{r!}\, x^{i_1}_{a_1}\cdots x^{i_r}_{a_r}\Big[(\partial/\partial t_a)B^{aa_1\cdots a_r}_{ii_1\cdots i_r} - B^{a_1\cdots a_r}_{i_1\cdots i_r i}\Big] = 0.$$

Since these equations must hold for (t,x) in R_0 and x_t arbitrary, the equations (2:2) follow at once, thus completing the proof of the necessity of the conditions stated in the theorem.

When the integrand function f has the form described in the theorem, we let S_1 and S_2 be two admissible m-spaces in R_0 with the same bounding $(m-1)$-space L and defined by the equations

$$x^i = x^{1i}(t), \qquad x^i = x^{2i}(t) \qquad (t \text{ in } T).$$

The function $z(t)$ will be defined by the equations

$$(2:7) \qquad z^i(t) = x^{2i}(t) - x^{1i}(t).$$

The family of admissible m-spaces

$$x^i = x^{1i}(t) + ez^i(t)$$

then contains S_1 for $e = 0$ and S_2 for $e = 1$. The derivative of the m-tuple integral

$$I(e) = \int_T f(t, x + ez, x_t + ez_t)\,dt$$

with respect to e has the value

$$I'(e) = \int_T (z^i f_{x^i} + z^i_a f_{x^i_a})\,dt.$$

Since the Euler equations (1:3) for the integrand function f are by hypothesis identically satisfied it follows that this derivative has also the value

$$I'(e) = \int_T [d(z^i f_{x_a^i})/dt_a]dt.$$

By means of an extended Green's Theorem[1], this last integral is equal, except perhaps for sign, to the $(m-1)$-tuple integral

$$(2{:}8) \qquad \int_A z^i f_{x_a^i} d_a t \qquad (d_a t \equiv dt_1 \cdots dt_{a-1}dt_{a+1} \cdots dt_m).$$

Since the two admissible spaces S_1 and S_2 have the same boundary L, equation (2:7) shows that z^i vanishes identically over A and hence the value of the integral (2:8) is zero. Thus I(0) and I(1) are equal, and since S_1 and S_2 can be selected arbitrarily from the class of admissible m-spaces in R_0 having the same fixed $(m-1)$-space L as boundary, the theorem is proved.

In order to show more clearly the nature of the conditions (2:2), we shall write them in more detail for a few special cases.

When $s = 2$, the conditions (2:2) become

$$(2{:}9) \qquad \partial B_i^a/\partial t_a = \partial B/\partial x^i$$

$$(2{:}10) \qquad \partial B_{ij}^{ab}/\partial t_a = \partial B_j^b/\partial x^i - \partial B_i^b/\partial x^j$$

$$(2{:}11) \qquad \partial B_{jk}^{ab}/\partial x^i + \partial B_{ki}^{ab}/\partial x^j + \partial B_{ij}^{ab}/\partial x^k = 0$$

If $m = 2$ and $n > 2$, these conditions may be written

$$\partial B_i^1/\partial t_1 + \partial B_i^2/\partial t_2 = \partial B/\partial x^i$$

$$\partial B_{ij}^{12}/\partial t_1 = \partial B_j^2/\partial x^i - \partial B_i^2/\partial x^j, \qquad \partial B_{ij}^{12}/\partial t_2 = \partial B_i^1/\partial x^j - \partial B_j^1/\partial x^i$$

$$\partial B_{jk}^{12}/\partial x^i + \partial B_{ki}^{12}/\partial x^j + \partial B_{ij}^{12}/\partial x^k = 0$$

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[1] P. Franklin, Multiple integrals in n-space, Annals of Mathematics, Second Series, vol. 24 (1922-23), p. 217.

If n = 2 and m $\geqq$ 2, the remarks below equations (2:6) show that equations (2:11) are always satisfied. In this case, equations (2:9) and (2:10) may be written

$$\partial B_1^a/\partial t_a = \partial B/\partial x^1, \qquad \partial B_2^a/\partial t_a = \partial B/\partial x^2$$

$$\partial B_{12}^{ab}/\partial t_a = \partial B_2^b/\partial x^1 - \partial B_1^b/\partial x^2 .$$

In particular, in this last case when m = 2, we have

$$\partial B_1^1/\partial t_1 + \partial B_1^2/\partial t_2 = \partial B/\partial x^1, \qquad \partial B_2^1/\partial t_1 + \partial B_2^2/\partial t_2 = \partial B/\partial x^2$$

$$\partial B_{12}^{12}/\partial t_2 = \partial B_1^1/\partial x^2 - \partial B_2^1/\partial x^1, \qquad \partial B_{12}^{12}/\partial t_1 = \partial B_2^2/\partial x^1 - \partial B_1^2/\partial x^2.$$

When s = 3, the conditions (2:2) become

$$\partial B_i^a/\partial t_a = \partial B/\partial x^i$$

$$\partial B_{ij}^{ab}/\partial t_a = \partial B_j^b/\partial x^i - \partial B_i^b/\partial x^j$$

$$\partial B_{ijk}^{abc}/\partial t_a = \partial B_{jk}^{bc}/\partial x^i + \partial B_{ki}^{bc}/\partial x^j + \partial B_{ij}^{bc}/\partial x^k$$

$$\partial B_{jkh}^{abc}/\partial x^i - \partial B_{khi}^{abc}/\partial x^j + \partial B_{hij}^{abc}/\partial x^k - \partial B_{ijk}^{abc}/\partial x^h = 0.$$

3. Investigation of fields by the use of specialized coordinates. Let the equations

$$(3{:}1) \qquad x^i = X^i(t,c) \qquad [(t,c) \text{ in } G]$$

be a family of extremals depending upon n parameters $c = (c^1, \ldots, c^n)$ such that the functions X^i, X_a^i belonging to the family have continuous partial derivatives of the third order at points (t,c) in a region G whose projection in $(t_1, \ldots, t_m)$-space is convex. The extremals (3:1) are said to simply cover a region F of points (t,x) if to each point of the region there corresponds one and only one solution (t,c) of equations (3:1) in the region G, and if furthermore the determinant $|X^i_{c^j}|$ is different from zero at each such solution. These solutions then define a set of single-valued functions $c^i(t,x)$ which have continuous partial derivatives of the third order in F. The functions

$$p^i_a(t,x) = X^i_a[t,c(t,x)]$$

are called the slope functions of the family in F.

By placing

$$y^i = c^i \tag{3:2}$$

in equations (3:1) we have a transformation

$$x^i = X^i(t,y) \tag{3:3}$$

by which the coordinates (t,x) are changed to (t,y). Differentiation with respect to t_a of these equations with $y^i = y^i(t)$ gives

$$x^i_a = X^i_a + X^i_{y^j} y^j_a. \tag{3:4}$$

Under the transformation (3:3) let

$$f(t,x,x_t) = g(t,y,y_t), \qquad D(t,x,x_t) = D^*(t,y,y_t) \tag{3:5}$$

where x_t and y_t are connected by the relations (3:4). Equations (1:1) and (1:4) become respectively

$$I = \int_T g(t,y,y_t)dt \tag{3:6}$$

$$I^* = \int_T D^*(t,y,y_t)dt. \tag{3:7}$$

The family of extremals (3:1) is now replaced by equations (3:2) and the slope functions of F become

(3:8) $$q^i_a(t,y) \equiv 0.$$

The Euler equations are

$$dg_{y^i_a}/dt_a = g_{y^i} ,$$

which can be written

$$g_{y^i_a t_a} + y^j_a g_{y^i_a y^j} + y^j_{ba} g_{y^i_a y^j_b} = g_{y^i}.$$

When the family of extremals (3:2) is substituted in these equations one obtains the relations

(3:9) $$g_{y^i_a t_a}(t,y,0) = g_{y^i}(t,y,0).$$

In order that the region F be a field, we must have an invariant integral of the form (3:7) with the integrand satisfying the equations

(3:10) $$D^*(t,y,0) = g(t,y,0), \qquad D^*_{y^i_a}(t,y,0) = g_{y^i_a}(t,y,0).$$

From Theorem 2:1 we see that the integrand of this invariant integral must be of the form

$$D^*(t,y,y_t) = C(t,y) + \sum_{r=1}^{s} \frac{1}{r!} y^{i_1}_{a_1} \cdots y^{i_r}_{a_r} C^{a_1 \cdots a_r}_{i_1 \cdots i_r}(t,y)$$

with the C's satisfying the equations

(3:11) $$(\partial/\partial t_a) C^{aa_1 \cdots a_r}_{ii_1 \cdots i_r} = C^{a_1 \cdots a_r}_{i_1 \cdots i_r i} \qquad (r = 0,1,\ldots,s),$$

where analogues of the identities (2:3) and all properties of the B's hold for the C's. Using equations (3:10) we have

(3:12) $$C(t,y) = D^*(t,y,0) = g(t,y,0), \quad C^a_i(t,y) = D^*_{y^i_a}(t,y,0) = g_{y^i_a}(t,y,0).$$

The integrand function D^* becomes

$$(3{:}13)\quad D^*(t,y,y_t)=g(t,y,0)+y^i_a g_{y^i_a}(t,y,0)+\sum_{r=2}^{s}\frac{1}{r!}y^{i_1}_{a_1}\cdots y^{i_r}_{a_r}C^{a_1\ldots a_r}_{i_1\ldots i_r}(t,y)$$

and the conditions given by (3:11) are

$$(3{:}14)\qquad g_{y^i_a t_a}(t,y,0) = g_{y^i}(t,y,0),$$

$$(3{:}15)\qquad (\partial/\partial t_a)C^{aa_1\ldots a_r}_{ii_1\ldots i_r} = C^{a_1\ldots a_r}_{i_1\ldots i_r i} \qquad (r = 1,\ldots,s).$$

Equations (3:14) are identical with equations (3:9) and by hypothesis are satisfied.

We shall now prove two theorems the first of which is the following:

THEOREM 3:1. *If* $m > 1$, *every region* F *of* tx-*space which is simply covered by an* n-*parameter family of extremals, in the sense described above, has associated with it an invariant integral* I^* *with which it forms a field for the integral* I *specified in* (1:1). *If the coordinates are transformed as described above the extremals of the field will be given by equations* (3:2) *and the slope functions by the equations* (3:8). *There will be an invariant integral* I^* *of the field in which the integrand function* D_1^* *is*

$$(3{:}16)\qquad D_1^*(t,y,y_t) = g(t,y,0) + y^i_a g_{y^i_a} + \frac{1}{2!}L^{ab}_{ij}y^i_a y^j_b\ ,$$

where

$$(3{:}17)\qquad \begin{aligned} mL^{ab}_{ij} &= \int_{\bar{t}_a}^{t_a}[K^b_{ji} + \frac{1}{m-1}\overline{K}^b_{ji}]dt_a + \int_{\bar{t}_b}^{t_b}[K^a_{ij} + \frac{1}{m-1}\overline{K}^a_{ij}]dt_b\ ,\\ K^a_{ij} &= g_{y^i_a y^j}(t,y,0) - g_{y^j_a y^i}(t,y,0), \end{aligned}$$

and $\overline{K}^a_{ij}$ *is obtained from* K^a_{ij} *by replacing* t_a *by* $\bar{t}_a$. *Furthermore, the integrand* D^* *of every invariant integral* I^* *associated with* F *must be of the form* (3:13) *with*

$$(3{:}18)\qquad C^{a_1\ldots a_r}_{i_1\ldots i_r} = K^{a_1\ldots a_r}_{i_1\ldots i_r} + L^{a_1\ldots a_r}_{i_1\ldots i_r} \qquad (r = 2,\ldots,s).$$

Here the functions $K^{a_1\ldots a_r}_{i_1\ldots i_r}$ *are solutions of the equations*

$$\frac{\partial}{\partial t_a} K^{a\,a_1\dots a_r}_{i_1 i_2\dots i_r} = 0 \qquad (r = 2,\dots,s), \tag{3:19}$$

$$K^{a_1\dots a_s}_{i_1\dots i_s i} = 0 \tag{3:20}$$

with the property that the interchange of two a's *or two* i's *simply changes the sign. Here*

$$\begin{aligned} K^{a_1\dots a_r}_{i_1\dots i_r i} = \frac{\partial}{\partial y^i} K^{a_1\dots a_r}_{i_1\dots i_r} - \frac{\partial}{\partial y^{i_1}} K^{a_1 a_2\dots a_r}_{i\;i_2\dots i_r} - \dots \\ - \frac{\partial}{\partial y^{i_r}} K^{a_1\dots a_{r-1} a_r}_{i_1\dots i_{r-1} i}\;. \end{aligned} \tag{3:21}$$

The function L^{ab}_{ij} *is defined by* (3:17) *and when* $r \geqq 2$ *the function* $L^{aa_1\dots a_r}_{ii_1\dots i_r}$ *is defined by the formula*

$$\begin{aligned} mL^{aa_1\dots a_r}_{ii_1\dots i_r} = \int_{\bar{t}_a}^{t_a} L^{a_1\dots a_r}_{i_1\dots i_r i}\, dt_a - \int_{\bar{t}_{a_1}}^{t_{a_1}} L^{a\;a_2\dots a_r}_{i_1 i_2\dots i_r i}\, dt_{a_1} \\ - \dots - \int_{\bar{t}_{a_r}}^{t_{a_r}} L^{a_1\dots a_{r-1} a}_{i_1\dots i_{r-1} i_r i}\, dt_{a_r}\;, \end{aligned} \tag{3:22}$$

where

$$L^{a_1\dots a_r}_{i_1\dots i_r i} = K^{a_1\dots a_r}_{i_1\dots i_r i} + \sum_{\tau^k} \frac{k!}{(m-1)\cdots(m-k)} K^{a_1\dots a_r}_{i_1\dots i_r i}(\tau^k), \tag{3:23}$$

the function $K^{a_1\dots a_r}_{i_1\dots i_r i}(\tau^k)$ *is obtained from* $K^{a_1\dots a_r}_{i_1\dots i_r i}$ *by replacing* k *of the values* $t_{a_1}, \dots, t_{a_r}$ *by the corresponding values* $\bar{t}_{a_1}, \dots, \bar{t}_{a_r}$ *and the sum is taken over all possible replacements* τ^k *of this type in which* $k \leqq r$.

Here and elsewhere the a's used in the limits of integration are not to be summed.

In order to prove the first part of the theorem we must show that the conditions expressed by equations (3:15) are satisfied by the coefficients $C^{ab}_{ij} = L^{ab}_{ij}$, $C^a_{ij} = K^a_{ij}$, $C^{a_1\dots a_r}_{i_1\dots i_r} = 0$ in equations (3:16). These conditions take the special form

$$\frac{\partial}{\partial t_a} L^{ab}_{ij} = K^b_{ji}\;, \tag{3:24}$$

$$\frac{\partial L^{ab}_{ij}}{\partial y^k} - \frac{\partial L^{ab}_{kj}}{\partial y^i} - \frac{\partial L^{ab}_{ik}}{\partial y^j} = 0. \tag{3:25}$$

The equation (3:25) holds with L^{ab}_{ij} replaced by K^{a}_{ij}, as is seen by the use of the definition (3:17) of K^{a}_{ij}. In view of this fact it follows from the definition (3:17) of L^{ab}_{ij} that equation (3:25) must also hold. To establish equation (3:24) we need the equations

$$(3:26) \qquad \frac{\partial}{\partial t_a} K^{a}_{ij} = 0, \qquad K^{a}_{ij} = - K^{a}_{ji} ,$$

which are easy consequences of equations (3:14) and (3:17). From equations (3:17) it is found that (a not summed)

$$m \frac{\partial L^{ab}_{ij}}{\partial t_a} = K^{b}_{ji} + \frac{1}{m-1} \overline{K}^{b}_{ji} + \int_{\bar{t}_b}^{t_b} \frac{K^{a}_{ij}}{t_a} \, dt_b \qquad (a \neq b),$$

$$m \frac{\partial L^{ab}_{ij}}{\partial t_a} = 0 \qquad (a = b).$$

Summing for a one obtains, by equations (3:26), the relation

$$m \frac{\partial L^{ab}_{ij}}{\partial t_a} = (m - 1)K^{b}_{ji} + \overline{K}^{b}_{ji} - [K^{b}_{ij} - \overline{K}^{b}_{ij}] = mK^{b}_{ji},$$

as was to be proved.

In order to establish the second part of Theorem 3:1 we observe that the equations

$$(3:27) \qquad \frac{\partial}{\partial t_a} K^{a\,a_2\ldots a_r}_{i_1 i_2 \ldots i_r i} = 0 \qquad (r \geqq 2),$$

$$(3:28) \quad \frac{\partial}{\partial y^k} K^{a_1\ldots a_r}_{i_1\ldots i_r i} - \frac{\partial}{\partial y^{i_1}} K^{a_1 a_2\ldots a_r}_{k i_2 \ldots i_r i} - \cdots - \frac{\partial K^{a_1\ldots a_r}_{i_1\ldots i_r k}}{\partial y^i} = 0$$

hold, by virtue of equations (3:19) and (3:21). Using equations (3:28) and (3:23) it is found that equation (3:28) holds when $K^{a_1\ldots a_r}_{i_1\ldots i_r i}$ is replaced by $L^{a a_1\ldots a_r}_{i i_1\ldots i_r}$. From this fact it follows that

$$C^{a_1\ldots a_r}_{i_1\ldots i_r i} = K^{a_1\ldots a_r}_{i_1\ldots i_r i} \qquad (r \geqq 2)$$

when equation (3:18) holds, the right member being obtained by equation (3:21) by replacing K by C. By the use of this result and equations (3:18) and (3:19), the equations (3:15) reduce to

$$\frac{\partial}{\partial t_a} L^{aa_1\cdots a_r}_{ii_1\cdots i_r} = K^{a_1\cdots a_r}_{i_1\cdots i_r i} \qquad (r = 2,\cdots,s). \tag{3:29}$$

When the integers $a_1, \cdots, a_r$ are not distinct, the right and left members of this equation are automatically zero, in view of the symmetry properties of the K's and the L's relative to their upper indices. When $r = s$ the first member of (3:29) is zero since the upper or the lower indices cannot all be distinct in this case. The equation (3:29) then follows from equation (3:20). It remains to discuss the case when $r < s$ and the integers $a_1, \cdots, a_r$ are distinct. Since $L^{aa_1\cdots a_r}_{ii_1\cdots i_r} = 0$ when a is one of the values $a_1, \cdots, a_r$, the sum in (3:29) can be taken over the $m - r$ integers from 1 to m with $a_1, \cdots, a_r$ deleted. Summing over these integers it is found, by formula (3:22), that

$$\begin{aligned} m\frac{\partial}{\partial t_a} L^{aa_1\cdots a_r}_{ii_1\cdots i_r} = (m - r)L^{a_1\cdots a_r}_{i_1\cdots i_r i} &- \int_{\bar{t}_{a_1}}^{t_a} \frac{\partial}{\partial t_a} L^{a\,a_2\cdots a_r}_{i_1 i_2\cdots i_r i}dt_{a_1} \\ &- \cdots - \int_{\bar{t}_{a_r}}^{t_{a_r}} L^{a_1\cdots a_{r-1}a}_{i_1\cdots i_{r-1}i_r i}\,dt_{a_r}. \end{aligned} \tag{3:30}$$

In order to compute the value of the first integral in (3:30) we observe that in equation (3:27) the sum may be taken over the $m - r + 1$ values distinct from $a_2, \cdots, a_r$. Using this fact together with equations (3:23) and (3:27) and summing for a distinct from $a_1, \cdots, a_r$ we find by integration that

$$\begin{aligned} &\int_{\bar{t}_{a_1}}^{t_a} \frac{\partial L^{a\,a_2\cdots a_r}_{i_1 i_2\cdots i_r i}}{\partial t_a}\,dt_{a_1} = \\ &\quad - \left[K^{a_1\cdots a_r}_{i_1\cdots i_r i} + \sum_{\tau_1^k}\frac{k!}{(m-1)\cdots(m-k)}\,K^{a_1\cdots a_r}_{i_1\cdots i_r i}(\tau_1^k)\right]^{t_{a_1}}_{\bar{t}_{a_1}}, \end{aligned} \tag{3:31}$$

where τ_1^k denotes that k of the values $t_{a_2}, \cdots, t_{a_r}$ have been replaced by the corresponding values $\bar{t}_{a_2}, \cdots, \bar{t}_{a_r}$, and the sum is taken for all possible replacements τ_1^k of this type. By the use of formula (3:31) and similar ones for the remaining integrals in (3:30), it is found that the equation (3:30) reduces to (3:29), except for the factor m. The equation (3:29) therefore holds and the theorem is proved.

Corresponding to Theorem 3:1, when $m = 1$, we have the following theorem.

THEOREM 3:2. *If* $m = 1$, *every region* F *of the* tx-*space which is simply covered by an* n-*parameter family of extremals, in the sense described above, has associated with it an invariant integral* I^* *with which it forms a field for the integral* I *specified in* (1:1) *if and only if, when the coordinates are transformed as described above, the equations*

$$g_{y_1^i y^j}(t,y,0) = g_{y_1^j y^i}(t,y,0) \qquad (i,j = 1,\dots,n) \tag{3:32}$$

are satisfied. Then, the extremals of the field will be given by (3:2), *the slope functions by the equations* (3:8) *and there will be an invariant integral* I^* *of the field in which the integrand function* D^* *is*

$$D^*(t,y,y_t) = g(t,y,0) + y_1^i g_{y_1^i}(t,y,0). \tag{3:33}$$

If $m = 1$, equations (3:15) with $r = 2, \dots, s$ do not appear and, by means of equations (3:12), equations (3:15) with $r = 1$ reduce to equations (3:32), hence the theorem. In this theorem it is evident that equations (3:32) are always satisfied for $n = 1$.

4. *Results expressed in terms of the original coordinates.* In the last section we obtained results in terms of specialized coordinates (t,y), which are related to the coordinates (t,x) by the transformation (3:3). We shall use the inverse transformation

$$y^i = Y^i(t,x) \tag{4:1}$$

in expressing these results in terms of the original coordinates (t,x).

We shall now derive several sets of identities. Substituting equations (3:3) into relations (4:1), and vice versa, we find

(4:2) $$y^i = Y^i[t,X(t,y)], \qquad x^i = X^i[t,Y(t,x)].$$

Differentiating these identities with respect to t_a we have

(4:3) $$0 = Y^i_a + Y^i_{x^j}p^j_a, \qquad 0 = X^i_a + X^i_{y^j}Y^j_a.$$

Differentiating the first group of identities (4:2) with respect to y^j and the second group with respect to x^j we obtain

(4:4) $$\delta^i_j = Y^i_{x^k}X^k_{y^j}, \qquad \delta^i_j = X^i_{y^k}Y^k_{x^j},$$

where $\delta^i_j = 1$ when $i = j$ and $\delta^i_j = 0$ for $i \neq j$.

Differentiation with respect to t_a of equation (4:1) with $x^j = x^j(t)$ gives

$$y^i_a = Y^i_a + Y^i_{x^j}x^j_a .$$

Using the first group of identities (4:3), these equations become

(4:5) $$y^i_a = (x^j_a - p^j_a)Y^i_{x^j}.$$

By means of equations (3:3), (3:4) and (3:5) we find

(4:6) $$g(t,y,y_t) = f(t,X,X_t + X_y y_t).$$

Making use of equations (3:1), (3:2) and (4:6), we obtain the relations

(4:7) $$g(t,y,0) = f(t,x,p), \qquad g_{y^i_a}(t,y,0) = X^j_{y^i}f_{x^j_a}(t,x,p).$$

From relations (4:4), (4:5) and (4:7) we find

(4:8) $$y^i_a g_{y^i_a}(t,y,0) = (x^i_a - p^i_a)f_{x^i_a}(t,x,p).$$

Under the transformation (4:1) let us set

$$(4{:}9)\quad C^{a_1\ldots a_r}_{i_1\ldots i_r}(t,Y) = X^{j_1}_{y^{i_1}} \ldots X^{j_r}_{y^{i_r}} A^{a_1\ldots a_r}_{j_1\ldots j_r}(t,x) \quad (r = 2,\ldots,s).$$

By means of equations (3:5), (4:4), (4:7), (4:8) and (4:9), we now have the following lemma.

LEMMA 4:1. *The integrand function* D *of an invariant integral* I^* *of the form* (1:4) *must have the form*

$$(4{:}10)\quad \begin{aligned} D(t,x,x_t) &= f(t,x,p) + (x^i_a - p^i_a) f_{x^i_a}(t,x,p) \\ &+ \sum_{r=2}^{s} \frac{1}{r!}(x^{i_1}_{a_1} - p^{i_1}_{a_1}) \ldots (x^{i_r}_{a_r} - p^{i_r}_{a_r}) A^{a_1\ldots a_r}_{i_1\ldots i_r}(t,x), \end{aligned}$$

where

$$A^{a_1\ldots a_r}_{i_1\ldots i_r}(t,x) = Y^{j_1}_{x^{i_1}} \ldots Y^{j_r}_{x^{i_r}} C^{a_1\ldots a_r}_{j_1\ldots j_r}(t,Y).$$

On transforming back to the original coordinates (t, x), Theorem 3:1 becomes

THEOREM 4:1. *If* m > 1, *every region* F *of* tx-*space which is simply covered by an* n-*parameter family of extremals* (3:3), *in the sense described at the beginning of Section* 3, *has associated with it an invariant integral* I^* *with which it forms a field for the integral* I *specified in* (1:1). *The slope functions of the field will be given by the equations*

$$(4{:}11)\qquad p^i_a(t,x) = X^i_a[t,Y(t,x)].$$

There will be an invariant integral I^* *of the field in which the integrand function* D *is of the form* (4:10) *with the* A's *for which* r > 2 *identically zero and*

$$A^{ab}_{ij} = Y^k_{x^i} Y^h_{x^j} L^{ab}_{kh}$$

where L^{ab}_{ij} *is given by formula* (3:17) *with*

$$K^a_{ij} = \frac{\partial}{\partial y^j}\,[X^k_{y^i} f_{x^k_a}(t,x,p)] - \frac{\partial}{\partial y^i}\, f\,[X^h_{y^j} f_{x^h_a}(t,x,p)].$$

Furthermore the integrand of every invariant integral I^* *associated with* F *must be of the form* (4:10) *with*

$$A^{ab}_{ij} = Y^{k}_{x^{i}} Y^{h}_{x^{j}} [K^{ab}_{kh} + L^{ab}_{kh}]$$

and for $r = 3, \cdots, s$

$$A^{a_1 \cdots a_r}_{i_1 \cdots i_r} = [Y^{j_1}_{x^{i_1}} \cdots Y^{j_r}_{x^{i_r}}][K^{a_1 \cdots a_r}_{j_1 \cdots j_r} + L^{a_1 \cdots a_r}_{j_1 \cdots j_r}],$$

where the K's *and the* L's *are the functions defined in Theorem* 3:1.

Here, as elsewhere, i, j, k and h, with or without subscripts, range over the integers 1 to n. Similarly a and b range from 1 to m. Any of these letters used in indicating limits of integration are not to be summed.

By means of relations (4:7), the equations (3:32) become

$$X^{k}_{y^{i}}(\partial/\partial y^{j}) f_{x^{k}_{1}}(t,x,p) = X^{h}_{y^{j}}(\partial/\partial y^{i}) f_{x^{h}_{1}}(t,x,p).$$

These relations may be written

$$X^{k}_{y^{i_1}} X^{k_1}_{y^{j_1}}(\partial/\partial x^{k_1}) f_{x^{h}_{1}}(t,x,p) = X^{h}_{y^{j_1}} X^{h_1}_{y^{i_1}}(\partial/\partial x^{h_1}) f_{x^{h}_{1}}(t,x,p).$$

On multiplying by $Y^{i_1}_{x^{i}} Y^{j_1}_{x^{j}}$ and using the relations (4:4) we have

$$(\partial/\partial x^{j}) f_{x^{i}_{1}}(t,x,p) = (\partial/\partial x^{i}) f_{x^{j}_{1}}(t,x,p). \tag{4:12}$$

Theorem 3:2 can now be stated in the following manner.

THEOREM 4:2. *If* m = 1 *every region* F *of* tx-*space which is simply covered by an* n-*parameter family of extremals* (3:3), *in the sense described at the beginning of* Section 3, *has associated with* *it an invariant integral* I^* *with which it forms a field for the integral* I *specified in* (1:1) *if and only if the equations* (4:12) *are satisfied. Then, the slope functions of the field will be given by the equations* (4:11) *and there will be an invariant integral* I^* *of the field in which the integrand function* D *is of the form* (4:10) *where all of the* A's *are identically zero.*

5. *Certain systems of partial differential equations.* We shall first consider a determinant of the form

$$(5{:}1)\qquad \begin{vmatrix} \partial/\partial t_a \\ \partial U^q/\partial t_a \end{vmatrix} \qquad (a = 1,\dots,m;\ q = 2,\dots,m),$$

where the U's are arbitrarily selected functions having continuous partial derivatives of the second order with respect to t. The elements $\partial/\partial t_a$ of the first row are differential operators and they operate on the elements of their cofactors according to the usual rules for differentiating a determinant or a product. We shall prove the following lemma.

LEMMA 5:1. *Any determinant of the form* (5:1) *has the value zero.*

To prove this lemma, let us expand the determinant in terms of elements of the first row. The determinant (5:1) is then equal to

$$\sum_{a=1}^{m}(-1)^{a-1}(\partial/\partial t_a)\left|\partial U^q/\partial t_1 \dots \partial U^q/\partial t_{a-1}\ \ \partial U^q/\partial t_{a+1} \dots \partial U^q/\partial t_m\right|.$$

By operating with the element $\partial/\partial t_a$ on the columns of its minor, in accordance with the usual rules for differentiating a determinant, then moving the column on which we have operated over the ones which precede it, the value of the determinant (5:1) is a sum of terms of the form.

$$M = (-1)^{a-1+b-1}\left[\partial^2 U^q/(\partial t_b\,\partial t_a) \qquad \partial U^q/\partial t_e\right] \qquad \text{for } b < a,$$

plus a sum of terms of the form

$$N = (-1)^{a-1+b-2}\left[\partial^2 U^q/(\partial t_b\,\partial t_a) \qquad \partial U^q/\partial t_e\right] \qquad \text{for } b > a,$$

with e ranging over the integers 1, ..., m excluding a and b. For every possible pair of values of a and b giving a term M there is a corresponding term N obtained by interchanging the values of a and b, and conversely. Moreover, these corresponding values of M and N are equal but opposite in sign. Hence the value of the determinant (5:1) is zero and the lemma is proved.

We shall now prove the following theorem.

THEOREM 5:1. *The equations*

$$(5{:}2) \qquad (\partial/\partial t_a)K^{a a_2 \cdots a_r} = 0 \qquad (a_2 < \cdots < a_r;\ r = 2, \ldots, s < m),$$

in which the interchange of two a's *simply changes the sign of the* K's, *are satisfied if we set*

$$(5{:}3) \qquad K^{a_1 \cdots a_r} \qquad (a_1 < \cdots < a_r)$$

equal to $(-1)^{a_1 + \cdots + a_r}$ *multiplied by the determinant of the matrix which remains when the rows* $a_1, \ldots, a_r$ *are deleted from the* $(m - r)$*-rowed matrix*

$$\left\|\partial U^v/\partial t_a\right\| \qquad (a = 1, \ldots, m;\ v = r+1, \ldots, m),$$

where the U's *are arbitrarily selected functions having continuous partial derivatives of the second order with respect to* t.

Let us consider the determinant

$$(5{:}4)\qquad \begin{vmatrix} \partial/\partial t_a \\ V^d_a \\ \partial U^v/\partial t_a \end{vmatrix} \qquad (a = 1, \ldots, m;\ d = 2, \ldots, r;\ v = r+1, \ldots, m),$$

where the V's are arbitrarily selected functions independent of t. This determinant has the value zero, since it is of the form (5:1) with $U^d = V^d_a t_a$. We shall denote by $V_{a_2 \ldots a_r}$ the determinant formed by selecting the columns $a_2, \ldots, a_r$ from the matrix of V's where $a_2 < \ldots < a_r$. When the elements of the first row operate on the V's the result is zero since the V's are independent of t.

We shall now expand the determinant (5:4) in terms of minors of the first r rows and then expand these r-rowed minors in terms of their first rows. Using the functions (5:3) the value of the above determinant, except perhaps for sign, becomes

$$\begin{aligned} &[V_{a_2 \ldots a_r}(\partial/\partial t_{a_1}) + \ldots + (-1)^{r-1} V_{a_1 \ldots a_{r-1}}(\partial/\partial t_{a_r})] K^{a_1 \ldots a_r} \\ &= V_{a_2 \ldots a_r}(\partial/\partial t_{a_1}) K^{a_1 \ldots a_r} + \ldots + V_{a_1 \ldots a_{r-1}}(\partial/\partial t_{a_r}) K^{a_r a_1 \ldots a_{r-1}} \\ &= V_{a_2 \ldots a_r}(\partial/\partial t_a) K^{a a_2 \ldots a_r}, \end{aligned}$$

where in these summations a without a subscript is not restricted but the a's with subscripts range only over sets of values satisfying the conditions $a_1 < \ldots < a_r$. Thus we see that the coefficients of the determinants $V_{a_2 \ldots a_r}$, whose elements are arbitrarily selected functions independent of t, are the left sides of equations (5:2) with the K's having the values given by the functions (5:3). Since the value of the determinant (5:4) is zero, as observed above, and since the V's are arbitrary, it follows that equations (5:2) are satisfied by the functions (5:3) and the theorem is proved.

We shall now determine K's which will satisfy the conditions of Theorem 3:1. Since the interchange of two a's or two i's simply changes the sign all of the K's with indices $a_1,\dots,a_r$ and $i_1,\dots,i_r$ are completely determined by those for which $a_1 < \dots < a_r$ and $i_1 < \dots < i_r$, all of the equations (3:19) are satisfied when those for which $a_2 < \dots < a_r$ and $i_1 < \dots < i_r$ are satisfied and all of the equations (3:20) are satisfied when those for which $a_1 < \dots < a_s$ and $i < i_1 \dots < i_s$ are satisfied.

It is convenient to consider first K's which satisfy equations (3:19) when $s < m$. In this case $s = n$ and it can readily be shown from the properties of the K's that equations (3:20) are always satisfied. The equations (3:19) when $s < m$ will be satisfied by K's chosen according to the following theorem.

THEOREM 5:2. *The equations*

$$(5:5) \qquad (\partial/\partial t_a)K^{a\,a_2\dots a_r}_{i_1 i_2\dots i_r} = 0 \qquad (a_2 < \dots < a_r;\ i_1 < \dots < i_r;\ r = 2,\dots,s < m),$$

in which the interchange of two a's *or two* i's *simply changes the sign of the* K's, *are satisfied if we set*

$$(5:6) \qquad K^{a_1\dots a_r}_{i_1\dots i_r} \qquad (a_1 < \dots < a_r;\ i_1 < \dots < i_r)$$

equal to $(-1)^{a_1+\dots+a_r}$ *multiplied by the determinant of the matrix which remains when the rows* $a_1,\dots,a_r$ *are deleted from the* $(m-r)$-*rowed matrix*

$$\left\|(\partial/\partial t_a)U^{v}_{i_1\dots i_r}\right\| \qquad (v = r+1,\ \dots,\ m)$$

where the U's *are arbitrarily selected functions having continuous partial derivatives of the second order with respect to* t *and as many other continuous partial derivatives as are needed for the calculus of variations problem under consideration.*

For a fixed set of values of $i_1, \dots, i_r$ the equations (5:5) are of the form (5:2) and the functions (5:6) are of the form (5:3). Hence, this theorem follows at onse from Theorem 5:1.

In determining K's which will satisfy the conditions of Theorem 3:1 with $s = m = n$, as before, we observe that equations (3:20) are always satisfied. Equations (3:19) with $r = 2, \dots, s-1$ will be satisfied by K's chosen according to Theorem 5:2. We can select K's that will satisfy equations (3:19) with $r = s = m = n$ by means of the following theorem.

THEOREM 5:3. *When* $s = m = n$, *the equations*

$$(5{:}7) \qquad (\partial/\partial t_a)K^{a\,a_2\dots a_m}_{i_1 i_2\dots i_s} = 0 \qquad (a_2 < \dots < a_m;\ i_1 < \dots < i_s),$$

in which the interchange of two a's *or two* i's *simply changes the signs of the* K's, *are satisfied if we set*

$$(5{:}8) \qquad K^{1\cdots m}_{i_1\dots i_s} = V_{i_1\dots i_s} \qquad (i_1 < \dots < i_s),$$

where the V's *are arbitrarily selected functions independent of* t *having continuous partial derivatives of as many orders as are needed for the calculus of variations problem under consideration.*

This theorem follows at once since the left sides of equations (5:7) all vanish when the K's from equation (5:8) are substituted.

We shall now consider K's which will satisfy the conditions of Theorem 3:1 when $s = m < n$. Equations (3:19) with $r = 2, \dots, s-1$ will be satisfied by K's chosen according to Theorem 5:2. Equations (3:19) with $r = s = m$ will be satisfied by K's determined by means of Theorem 5:3 but we must make these same K's satisfy equations (3:20). This can be done by the use of the following theorem.

THEOREM 5:4. *When* $s = m < n$, *the equations*

$$(\partial/\partial y^{i})V_{i_1\ldots i_s} - (\partial/\partial y^{i_1})V_{i i_2\ldots i_s} - \ldots - (\partial/\partial y^{i_s})V_{i_1\ldots i_{s-1} i} = 0 \qquad (i < i_1 < \ldots < i_s), \tag{5:9}$$

where the interchange of two i's *simply changes the sign of the* V's, *are satisfied if we set*

$$V_{i_1\ldots i_s} \qquad (i_1 < \ldots < i_s) \tag{5:10}$$

equal to the determinant of the matrix obtained by selecting the rows $i_1, \ldots, i_s$ *from the* s-*rowed matrix*

$$\left\| (\partial/\partial y^j)W^u_{i_1\ldots i_s} \right\| \qquad (j = 1,\ldots,n;\ u = 1,\ldots,s),$$

where the W's *are arbitrarily selected functions independent of* t *having continuous partial derivatives of the second order with respect to* y *and as many other continuous partial derivatives as are needed for the calculus of variations problem under consideration.*

To prove this theorem let us consider a determinant of the form

$$\begin{vmatrix} \partial/\partial y^f \\ (\partial/\partial y^f)W^u_{i_1\ldots i_s} \end{vmatrix} \qquad \begin{pmatrix} f = i, i_1, \ldots, i_s; \\ u = 1, \ldots, s \end{pmatrix}. \tag{5:11}$$

This determinant is of the form (5:1), with W's and y's in place of U's and t's respectively, and hence it has the value zero.

Let us expand the determinant (5:11) in terms of elements of the first row. Making use of the functions (5:10), the determinant becomes

$$(\partial/\partial y^{i})V_{i_1 \ldots i_s} - (\partial/\partial y^{i_1})V_{ii_2 \ldots i_s} + \ldots + (-1)^s(\partial/\partial y^{s})V_{ii_1 \ldots i_{s-1}}$$

We see that these expressions and the left sides of equations (5:9) are equal, since the interchange of two i's simply changes the sign of the V's. As observed above, the determinant (5:11) has the value zero and hence the functions (5:10) satisfy equations (5:9), thus proving the theorem.

The conditions which the K's in Theorem 4:1 must satisfy are the same as those which the K's in Theorem 3:1 must satisfy. We have shown how these conditions can always be satisfied by choosing the K's in accordance with Theorem 5:2, Theorem 5:3 and Theorem 5:4.

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THE HAMILTON-JACOBI THEORY
FOR THE
PROBLEMS OF BOLZA AND MAYER

BY

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THE HAMILTON-JACOBI THEORY FOR THE PROBLEMS OF BOLZA AND MAYER

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1. Introduction. The problem of Bolza in non-parametric form is that of finding in a class of arcs

$$(1{:}1) \qquad y_i(t) \qquad (i = 1, \dots, n;\ t_1 \leqq t \leqq t_2)$$

satisfying the differential equations and end-conditions of the form

$$\phi_\alpha(t,y,y') = 0 \qquad (\alpha = 0,1,\dots,m < n-1),$$

$$\psi_\mu[t_1,y(t_1),t_2,y(t_2)] = 0 \qquad (\mu = 0,1,\dots,\ p \leqq 2n+1),$$

one which minimizes a sum of the form

$$J = g[t_1,y(t_1),t_2,y(t_2)] + \int_{t_1}^{t_2} f(t,y,y')dt.$$

The symbols y and y' represent the set $(y_1,y_2, \dots, y_n)$ and its derivative $(y_1',y_2', \dots, y_n')$.

In parametric form the problem is that of finding in a class of arcs

$$y_i(t) \qquad (i = 1, \dots, n;\ t_1 \leqq t \leqq t_2)$$

satisfying the differential equations and end-conditions of the form

$$(1{:}2) \qquad \phi_\beta(y,y') = 0 \qquad (\beta = 1, \dots, m < n-1),$$

$$\psi_\sigma[y(t_1), y(t_2)] = 0 \qquad (\sigma = 1, \ldots, p \leqq 2n), \tag{1:3}$$

one which minimizes a function of the form

$$J = g[y(t_1), y(t_2)] + \int_{t_1}^{t_2} f(y, y')dt. \tag{1:4}$$

The functions ϕ_β and the function f are assumed to be positively homogeneous of order one in the derivatives y' in order that the solutions of equations (1:2) and the integral in (1:4) will be independent of the parameter t.

The parametric problem in y-space may be considered as a non-parametric problem in ty-space if, to equations (1:2) and (1:3), there are adjoined, respectively, the equations

$$\phi_0(y, y') = \phi(y, y') - 1 = 0, \tag{1:5}$$

$$\psi_0 = t_1 = 0, \tag{1:6}$$

which define the parameter t uniquely. The function ϕ is a conveniently chosen function of the variables y and y', positively homogeneous of order one in y'.

These three formulations of the problem of Bolza will be referred to as problems A, B, C, *respectively*. The theory of problem B will be deduced from that of problem C, which is a special case of problem A.

The problem of Mayer with variable end-points in non-parametric form as formulated by Bliss is the problem A with its integrand function f identically zero. It reduces to the classical problem of Mayer when the expression to be minimized is the function $g = y_1(t_2)$ and the end-conditions $\psi_\mu = 0$ are $t_1 - \alpha_1 = y_{i1} - \beta_{i1} = t_2 - \alpha_2 = y_{j2} - \beta_{j2} = 0$, $(i = 1, \ldots, n;\ j = 2, \ldots, n)$ the α's and β's being constants. Similarly, the problem of Mayer with variable end-points in parametric form is the problem B with its integrand function f identically zero. It reduces to the classical problem of Mayer when the expression to be minimized is $g = y_1(t_2)$ and the end-conditions $\psi_\sigma = 0$ are $y_1(t_1) - \beta_{11} = y_j(t_2) - \beta_{j2} = 0$.

It is the purpose of this paper to extend the known Hamilton-Jacobi and field theory for the problem of Lagrange in parametric and non-parametric form to the problems of Bolza and Mayer in parametric and non-parametric form. In general, the theory for the parametric problem is deduced from that of the non-parametric problem by a method analogous to that used by Teach (X) Throughout the paper the emphasis is placed upon the Hamilton function H and, in particular, the field theory is based upon the definition of a field in terms of the canonical variables given in Section 6. The corresponding theory in terms of the function F defined in Section 3 is readily obtained with the aid of the definition of the function H given in Section 4.

The analytic basis of the problem is given in Section 2. The multiplier rule and its usual corollaries are given in Section 3 for both problems A and B. It is shown that along a minimizing arc for problem C the multiplier λ_0 associated with ϕ_0 has the value zero. This result agrees with that obtained by Teach for the problem of Lagrange and was deduced by the same method.

In Section 4 the imbedding theorem for the extremals of problem B is deduced from the corresponding theorem for problem A. The Hamilton function H is defined, and some of its properties are determined. The homogeneity of the function H for the problem of

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[*] Roman numerals in parentheses refer to the bibliography at the end of the paper.

Mayer in parametric form and its effect upon the imbedding theorem are deduced from the theory developed for the non-parametric problem by Bliss and Hestenes (III). The necessary conditions of Weierstrass and Clebsch are given in regular and canonical variables in Section 5.

In addition to the definition of a Mayer field for problem A in terms of regular variables due to Bliss, a second definition of a Mayer field in terms of the canonical variables and the Hamilton function H is given in Section 6. The field theory developed in this paper is based upon the second definition.

The next two sections are devoted to a discussion of transversal surfaces of a Mayer field for problem A, the construction of fields from a 2n-parameter family of extremals, and the relations between the extremals and a complete integral of the Hamilton-Jacobi equation. In Section 8 a necessary and sufficient condition for the non-appearance of some of the parameters in the functions defining the extremal arcs of a field is given in terms of the function defining the transversal surfaces of the field. This result enables us to obtain at once from the Hamilton-Jacobi theory for the problem of Bolza further results for the problem of Mayer and other problems whose extremals are abnormal on every subinterval.

Fields for the problem of Mayer have some unusual properties on account of the homogeneity of the Hamilton function H. The form which they take for the non-parametric problem is discussed in Section 9. In particular, it is shown that the extremals of a field lie on its transversal surfaces and that each transversal surface satisfies the definition of a field given by Kneser (IX, 43) for the problem of Mayer. A method is given for constructing an n-parameter family of Mayer fields from a 2n-parameter family of extremals. It is also shown how to determine a 2n-parameter family of extremals from a complete integral of the Hamilton-Jacobi equation. Finally, the theorem deduced in the previous section for problem A concerning the non-appearance of some of the parameters in the functions defining the extremal arcs of a field is applied to this problem.

Two definitions of a Mayer field for problem B are given in Section 10 which are analogous to those given in Section 6 for problem A. The field theory for problem B is based upon the definition of a field in terms of the canonical variables and the Hamilton function H. This section also includes the definition of the transversal surfaces of a field and a discussion of their properties.

The form which the field theory for problem A assumes for problem C is given in Section 11. The only fields for problem C which are important for problem B are those which are independent of the parameter t. It is shown how to construct such fields from a 2n-parameter family of extremals, using a method suggested by the proof of a similar theorem by Bliss for a parametric problem (V, 241). In Sections 12 and 13 the theory for problem B and the problem of Mayer in parametric form analogous to that developed in Sections 8 and 9 for the corresponding non-parametric problems is deduced from the theory developed in Section 11 for problem C.

The relation between the foregoing theory for the problem of Mayer in parametric and non-parametric form and the theory of Kneser (IX) is discussed in Sections 14 and 15. In particular, it is shown that a Mayer field as defined by Bliss consists of a one-parameter family of fields as defined by Kneser, and that every field of the latter type can be imbedded in a one-parameter family of such fields which form a field as defined by Bliss.

2. *Analytic basis for the problem*. The following definitions and assumptions will be made in order to carry through the analysis that follows:

For the problem A it is assumed that the functions ϕ_α and f have continuous partial derivatives of at least the first three orders on a $(2n+1)$-dimensional region D_1 of points (t, y, y') and that the matrix $\|\phi_{\alpha y'_i}\|$ has rank $m+1$ on this domain. The functions g, ψ_μ are assumed to have continuous second derivatives on a region D_2 of $(2n+2)$-dimensional points $[t_1, y(t_1), t_2, y(t_2)]$ and to be such that on this domain the matrix

$$\|\psi_{\mu t_1} \quad \psi_{\mu y_{i1}} \quad \psi_{\mu t_2} \quad \psi_{\mu y_{i2}}\|$$

has rank $p+1$. *An arc in* D_1 is an arc of type (1:1) which is continuous and consists of a finite number of sub-arcs on each of which the functions $y_i(t)$ have continuous derivatives and whose elements (t, y, y') are all interior to D_1. *An admissible arc* is an arc in D_1 with its end-element $|t_1, y(t_1), t_2, y(t_2)|$ interior to D_2. On every such arc the functions ϕ_α, ψ_μ, g and J have well-defined values.

For the problem B it is assumed that the functions ϕ_β and f have continuous partial derivatives of at least the first three orders and are positively homogeneous of order one in y' on a 2n-dimensional region $\mathcal{R}_1$ of points (y, y') which is such that if (y, y') is in $\mathcal{R}_1$, then (y, ky') for k positive is also in $\mathcal{R}_1$, and $y'_i y'_i \neq 0$. The matrix $\|\phi_{\beta y'_i}\|$ is assumed to have rank m on this domain. The functions g, ψ_σ are assumed to have continuous second derivatives on a region $\mathcal{R}_2$ of 2n-dimensional points $[y_{i1}, y_{i2}]$ and to be such that on this domain the matrix $\| \psi_{\sigma y_{i1}} \quad \psi_{\sigma y_{i2}} \|$ has rank p. The definitions of an arc in $\mathcal{R}_1$ and an admissible arc are analogous to those made for problem A.

For the problem C, which is a special form of problem A obtained from problem B as described above, the region $\mathfrak{D}_1$ corresponding to D_1 is taken to be the set of all elements (t, y, y') such that (y, y') is in $\mathcal{R}_1$. The function $\phi(y, y')$, appearing in ϕ_0, is chosen to be positive and to be positively homogeneous of order one in y' on $\mathfrak{D}_1$. Moreover, this function is selected so as to have continuous third derivatives and to be such that the extended matrix $\|\phi_{\alpha y_i'}\|$, $(\alpha = 0, 1, \ldots, m)$, has rank $m+1$ on $\mathfrak{D}_1$. For example, the function $\phi = (y_i' y_i')^{1/2}$ has these properties. The region $\mathfrak{D}_2$ is taken to be the set of all points $[t_1, y(t_1), t_2, y(t_2)]$ such that $[y(t_1), y(t_2)]$ is in $\mathcal{R}_2$.

Every admissible arc for problem B, satisfying the equations $\phi_\beta(y, y') = 0$ and the end-conditions $\psi_\sigma = 0$, can be changed uniquely by altering the parameter into one for problem C with the same properties, with $t_1 = 0$, and satisfying the equation $\phi_0 = \phi(y, y') - 1 = 0$. Hence, problem B is equivalent to problem C in the sense that there exists a one-to-one correspondence between their admissible arcs satisfying the differential equations and end-conditions such that corresponding arcs give to J the same value. It should be observed, however, that if E and E' represent corresponding arcs for problems B and C, respectively, then not every arc K in a neighborhood of E can be transformed into an arc K' lying in a prescribed neighborhood of E'. This is readily seen if the parameter t represents arc length since then the arc K can be chosen so that the length of K' exceeds that of E' by an arbitrary amount. The theory deduced for problem B is independent of the particular parameterization of the arcs used, but in this paper whenever the two problems B and C are being compared it will be understood that corresponding arcs have the parameterization determined by problem C.

A set ξ_1, ξ_2, $\eta_i(t)$, where ξ_1, ξ_2 represent constants, is called a set of admissible variations along an admissible arc E for problem A provided that $\eta_i(t)$ possesses the continuity properties of an admissible arc. The equations of variation of the functions ϕ_α, ψ_μ along E are given by the equations

$$\Phi_\alpha = \phi_{\alpha\, y_i}\eta_i + \phi_{\alpha\, y_i'}\eta_i' = 0 \qquad (\alpha = 0, 1, \ldots, m), \tag{2:1}$$

$$\Psi_\mu[\xi_1, \eta(t_1), \xi_2, \eta(t_2)] = 0 \qquad (\mu = 0, 1, \ldots, p), \tag{2:2}$$

where

$$\Psi_\mu = (\psi_{\mu t_1} + y_{i1}'\psi_{\mu y_{i1}})\xi_1 + \psi_{\mu y_{i1}}\eta_i(t_1)$$
$$+ (\psi_{\mu t_2} + y_{i2}'\psi_{\mu y_{i2}})\xi_2 + \psi_{\mu y_{i2}}\eta_i(t_2).$$

For problem C, $\Psi_0 = \xi_1 = 0$.

Similar definitions hold for problem B except that constants ξ_1, ξ_2 are not present. Equations (2:1) and (2:2) take the forms

$$\Phi_\beta = \phi_{\beta\, y_i}\eta_i + \phi_{\beta\, y_i'}\eta_i' = 0 \qquad (\beta = 1, \ldots, m), \tag{2:3}$$

$$\Psi_\sigma = \psi_{\sigma\, y_{i1}}\eta_i(t_1) + \psi_{\sigma\, y_{i2}}\eta_i(t_2) = 0 \quad (\sigma = 1, \ldots, p). \tag{2:4}$$

The functions $\eta_i(t)$ for problems B and C are understood to have the same parameterization. It should be noted that if E is an admissible arc for problem B defined by functions $y_i(t)$ having continuous second derivatives for $t_1 \leqq t \leqq t_2$ and if the functions $\eta_i(t)$ form a set of admissible variations satisfying the equations

(2:3) along E, then, on account of the homogeneity of the functions ϕ_β, the functions $\zeta_i(t) = \eta_i + \xi(t)y_i'(t)$, where $\xi(t)$ possesses the continuity properties of $\eta_i(t)$, are also solutions of equations (2:3). This property will be used in the proof of the following lemma.

LEMMA 2:1. *Let* E *be an admissible arc for problem* B *defined by functions* $y_i(t)$ *having continuous second derivatives on the interval* $t_1 \leqq t \leqq t_2$. *If* $\eta_i(t)$ *is a set of admissible variations satisfying the equations* (2:3) *along* E, *then* ξ_1, ξ_2, $\zeta_i(t) = \eta_i(t) - \xi(t)y_i'(t)$, *where*

$$\xi(t) = \int_{t_1}^{t} (\phi_{y_i}\eta_i + \phi_{y_i'}\eta_i')dt + c, \qquad \xi_1 = \xi(t_1), \qquad \xi_2 = \xi(t_2)$$

forms a set of admissible variations satisfying the equations (2:1) *along the corresponding arc* E' *for problem* C. *Conversely, if* ξ_1, ξ_2, $\zeta_i(t)$ *is a set of admissible variations satisfying equations* (2:1) *along* E', *then the set* $\eta_i(t) = \zeta_i(t) + \xi(t)y_i'$ *is a set of admissible variations satisfying the equations* (2:3) *along* E, *where* $\xi(t)$ *is an arbitrary function having* $\xi(t_1) = \xi_1$, $\xi(t_2) = \xi_2$ *and continuity properties like those of* $\zeta_i(t)$. *In either case* $\eta_i(t)$ *satisfy equations* (2:4) *if and only if the set* ξ_1, ξ_2, $\zeta_i(t)$ *satisfy the equations* (2:2) *for problem* C.

This result can be established by substitution with the help of the remark preceding this lemma.

An admissible arc E for problem A satisfying the equations $\phi_\alpha = 0$ is said to be *normal relative to the end-conditions*, and designated simply as *normal*, if there exist for it p + 1 sets of admissible variations ξ_{1v}, ξ_{2v}, $\eta_{iv}(t)$ satisfying equations (2:1) such that the determinant $|\Psi_\mu(\xi_v, \eta_v)|$ is different from zero for $(\mu, v = 0, 1, \ldots, p)$. Similarly, an admissible arc E for problem B satisfying the equations $\phi_\beta = 0$ is said to be normal if there exist for it p sets of admissible variations $\eta_{iq}(t)$ satisfying equations (2:3) such that the determinant $|\Psi_\sigma(\eta_q)|$ is different from zero for (σ, q = 1, ..., p).

LEMMA 2:2. Let E be an admissible arc for problem B defined by functions $y_i(t)$ having continuous second derivatives on the interval $t_1 \leqq t \leqq t_2$. Then E is normal if and only if its corresponding arc for problem C is normal.

Corresponding arcs for problems B and C have the same continuity properties since, as indicated previously, when the arcs are being compared they are assumed to have the same parameterization. If an admissible arc E' for problem C having the continuity properties stated in the lemma for the arc E is normal, a determinant of the form

$$\left|\Psi_{\sigma y_{i1}}[\zeta_{iv}(t_1) + \xi_{1v}y_i'(t_1)] \overset{\xi_{1v}}{} + \Psi_{\sigma y_{i2}}[\zeta_{iv}(t_2) + \xi_{2v}y_i'(t_2)]\right| \tag{2:5}$$

$$(v = 0, 1, \ldots, p;\ \sigma = 1, \ldots, p)$$

must be different from zero. It may be assumed without loss of generality that $\xi_{10} = 1$, $\xi_{1v} = 0$ $(v \neq 0)$. Then the determinant (2:5) is equal to the determinant

$$\left|\Psi_{\sigma y_{i1}}\zeta_{iq}(t_1) + \Psi_{\sigma y_{i2}}[\zeta_{iq}(t_2) + \xi_{2q}y_i'(t_2)]\right| \quad (\sigma, q = 1, \ldots, p).$$

But the sets $\eta_{iq} = \zeta_{iq}(t) + \xi_q(t)y_i'$, described in the second part of Lemma 2:1, are p sets of admissible variations satisfying equations (2:3) along the corresponding arc E for problem B. The arc E, consequently, is normal. Conversely, if the determinant $|\Psi_\sigma(\eta_{iq})|$ is different from zero for p sets of admissible variations $\eta_{iq}(t)$ satisfying the equations (2:3) along an arc E for problem B, then $\xi_{10} = 1$, $\xi_{20} = 0$, $\zeta_{10} = 0$ and ξ_{1q}, ξ_{2q}, ζ_{1q}, where the latter sets are related to η_{iq} as described in the first part of Lemma 2:1, are p+ 1 sets of admissible variations satisfying equations (2:1) along the corresponding arc E' for problem C for which the determinant (2:5) is different from zero. The arc E', therefore, is normal.

It will be assumed throughout this paper that the minimizing arc for the problem under discussion is normal.

3. *The multiplier rule*. From the theory of the problem of Bolza it is evident at once that a normal minimizing arc for problem A must satisfy the following multiplier rule with a unique set of multipliers (IV, 26).

THEOREM 3:1. *If* E *is a normal minimizing arc for problem* A, *there exist a unique set of constants* c, c_i, e_μ *and a unique function*

$$(3{:}1) \qquad F(t,y,y',\chi) = f + \chi_\alpha(t)\phi_\alpha \qquad (\alpha = 0,1,\dots,m)$$

such that the equations

$$(3{:}2) \quad F - y_i'F_{y_i'} = \int_{t_1}^{t} F_t\,dt + c, \qquad F_{y_i'} = \int_{t_1}^{t} F_{y_i}\,dt + c_i, \qquad \phi_\alpha(t,y,y') = 0$$

hold along E, *and furthermore such that the transversality condition*

$$(3{:}3) \qquad [(F - y_i'F_{y_i'})dt + F_{y_i'}\,dy_i]_1^2 + dg + e_\mu\,d\psi_\mu = 0$$

holds at the ends of E *for every choice of the differentials* dt_1, dt_2, dy_{i1}, dy_{i2}. *Along every subarc of* E *between corners the multipliers* $\chi_\alpha(t)$ *are continuous. If* $f \equiv 0$, *the multipliers* $\chi_\alpha(t)$ *are either identically zero or else do not vanish simultaneously at any point.*

The functions $F - y_i'F_{y_i'}$ and $F_{y_i'}$ are continuous along a minimizing arc E. On each subarc of E between corners they have continuous derivatives

$$(3{:}4) \qquad d(F - y_i'F_{y_i'})/dt = F_t, \qquad dF_{y_i'}/dt = F_{y_i}.$$

The equations

$$(3{:}5) \qquad dF_{y_i'}/dt = F_{y_i}, \qquad 0 = \phi_\alpha(t,y,y') \qquad \begin{array}{l}(i = 1, \ldots, n;)\\ (\alpha = 0, 1, \ldots, m)\end{array}$$

are called the *Euler-Lagrange equations*. If the arc E consists of a single arc without corners and with continuous derivatives $y_i''(t)$, $\chi_\alpha'(t)$, then the first equation in (3:4) is a consequence of equations (3:5) (IV,32). If the matrix

$$(3{:}6) \qquad R = \begin{Vmatrix} F_{y_i'y_k'} & \phi_{\gamma\, y_i'} \\ \phi_{\alpha\, y_k'} & 0 \end{Vmatrix} \qquad \begin{array}{l}(i,k = 1, \ldots, n)\\ (\alpha, \gamma = 0,1, \ldots,m)\end{array}$$

has rank $n+m+1$ at a point of a subarc of E between corners, then the functions $y_i'(t)$, $\chi_\alpha(t)$ belonging to the arc have continuous derivatives of at least the first order with respect to t at that point.

THEOREM 3:2. *For problem C, the equation*

$$(3{:}7) \qquad \chi_0 = y_i'\bar{F}_{y_i'} - \bar{F}$$

holds identically where $\bar{F}$ *is the function for problem* C *corresponding to the function* F *for problem* A *and* χ_0 *is the multiplier of* $\phi_0 = \phi - 1$ *in* $\bar{F}$. *Along a minimizing arc* E', *the multiplier* χ_0 *is equal to zero*.

The function $\bar{F}$ is given by the equation

$$\bar{F}(y,y',\lambda) = \lambda_0(\phi - 1) + F,$$

$$F(y,y',\lambda) = f(y,y') + \lambda_\beta(t)\phi_\beta(y,y') \qquad (\beta = 1, \ldots, m).$$

From the homogeneity relations $F(y, ky', \lambda) = kF(y, y', \lambda)$, $\phi(y, ky') = k\phi(y, y')$, where $k > 0$, it is found by differentiating with respect to k and setting $k = 1$ that

$$(3{:}8) \qquad F = y_i' F_{y_i'}, \qquad\qquad \phi = y_i'\phi_{y_i'} ,$$

and hence that $\bar{F} = \lambda_0(\phi - 1) + F = y_i'\bar{F}_{y_i'} - \lambda_0$. This proves the identity (3:7). Since $\bar{F}_t = 0$, the multiplier λ_0 has a constant value along a minimizing arc E' satisfying the equations $\phi_\alpha = 0$ and the first equation in (3:2). The transversality condition for E' requires that the coefficient of dt in (3:3), namely $-\lambda_0(t_2)$, must be zero. Hence, along E' the multiplier λ_0 is identically zero.

The function $\bar{F}$ along a minimizing arc E' for problem C is independent of the choice of ϕ and is equal to F. It is now possible to state the following multiplier rule for the corresponding arc E for problem B.

THEOREM 3:3. _If_ E _is_ _a_ _normal_ _minimizing_ _arc_ _for_ _problem_ B, _there_ _exist_ _unique_ _constants_ c_i, e_μ _and_ _a_ _unique_ _function_

$$F(y,y',\lambda) = f(y,y') + \lambda_\beta(t)\phi_\beta(y,y') \qquad (\beta = 1, \ldots, m)$$

such _that_ _the_ _equations_

$$F_{y_i'} = \int_{t_1}^{t} F_{y_i}\,dt + c_i, \qquad \phi_\beta(y,y') = 0$$

hold _along_ E, _and_ _furthermore_ _such_ _that_ _the_ _transversality_ _condition_

$$F_{y_i'}\,dy_i\Big|_1^2 + dg + e_\mu\,d\psi_\mu = 0$$

holds at the ends of E for every choice of the differentials dy_{i1}, dy_{i2}. *Along every subarc of E between corners the multipliers* $\chi_\beta(t)$ *are continuous and do not vanish simultaneously at any point.*

This result follows from Theorem 3:1, Theorem 3:2 and Lemma 2:2 only if the functions $y_i'(t)$, $\chi_\beta(t)$ belonging to the minimizing arc have continuous first derivatives. The theorem is, however, true as stated, but we shall not pause to establish it at this time.

The functions $F_{y_i'}$ are continuous along a minimizing arc E. On each subarc of E between corners they have continuous derivatives $dF_{y_i'}/dt = F_{y_i}$. The equations

$$(3{:}9) \qquad dF_{y_i'}/dt = F_{y_i}, \qquad 0 = \phi_\beta(y, y')$$

are called the *Euler-Lagrange equations* for problem B. The matrix for problem B, corresponding to the matrix R in (3:6) for problem A, is

$$(3{:}10) \qquad \mathcal{R} = \left\| \begin{matrix} F_{y_i' y_k'} & \phi_{\nu y_k'} \\ \phi_{\beta y_i'} & 0 \end{matrix} \right\| \qquad \begin{matrix} (i, k = 1, \ldots, n) \\ (\beta, \nu = 1, \ldots, m). \end{matrix}$$

Its rank is less than $n + m$ along an arc E satisfying the equations $\phi_\beta = 0$ because of the homogeneity of the functions F and ϕ_β and the identities

$$(3{:}11) \qquad y_i' F_{y_i' y_k'} = 0$$

which are obtained by differentiating each of the first n identities in (3:8) with respect to y_k'. Along an arc E' for problem C for which $\chi_0 = 0$, $\phi_\alpha = 0$, the matrix R in (3:6) assumes the form

(3:12) $$D = \left\| \begin{matrix} F_{y_i' y_k'} & \phi_{\gamma y_k'} \\ \phi_{\alpha y_i'} & 0 \end{matrix} \right\| \qquad \begin{matrix} (i,k = 1, \ldots, n) \\ (\alpha, \gamma = 0, 1, \ldots, m). \end{matrix}$$

THEOREM 3:4. *Let* E *be an arc for problem* B *satisfying the equations* $\phi_\beta = 0$ *which corresponds to an arc* E' *for problem* C *satisfying the equations* $\phi_\alpha = 0$ *and having the multiplier* λ_0 *equal to zero. Then if the matrix* $\mathcal{R}$ *in* (3:10) *is of rank* $n+m-1$ *along* E, *the matrix* D *in* (3:12) *is of rank* $n+m+1$ *along* E', *and conversely.*

If D has rank less than $n+m+1$, there exist constants λ_i, μ_α not all zero satisfying the equations

$$\begin{aligned} \lambda_i F_{y_i' y_k'} + \mu_\gamma \phi_{\gamma y_k'} &= 0 \\ \lambda_i \phi_{\alpha y_i'} &= 0. \end{aligned}$$

If each kth one of the first n equations is multiplied by y_k' and the n equations are added, the equation $\mu_0 = 0$ is obtained becuase of the conditions (3:8) and (3:11). It follows that the matrix

$$\left\| \begin{matrix} F_{y_i' y_k'} & \phi_{\nu y_k'} \\ \phi_{\alpha y_i'} & 0 \end{matrix} \right\| \qquad \begin{matrix} (i,k = 1, \ldots, n) \\ (\nu = 1, \ldots, m) \\ (\alpha = 0, 1, \ldots, m) \end{matrix}$$

must have rank less than $n+m$. Applying this same process to the columns of any matrix obtained from this one by deleting one of the first n or one of the last m rows, it is found that the rank of the matrix $\mathcal{R}$ must be less than $n+m-1$. On the other hand, if the rank of $\mathcal{R}$ is less than $n+m-1$, by adjoining one row and one column to obtain D, it is possible to increase the rank by at most two. Hence, the rank of D would have to be less than $n+m+1$.

THEOREM 3:5. *The differentiability condition. On each subarc of a normal minimizing arc* E *for problem* B *on which the functions* $y_i(t)$ *defining* E *have continuous derivatives and the matrix* $\mathcal{R}$ *in* (3:10) *has rank* $n+m-1$, *the functions* $y_i'(t)$, $\chi_\beta(t)$ *belonging to* E *have continuous derivatives of at least the first order with respect to* t.

This result follows from the differentiability condition for problem C by means of Theorem 3:4 and the last statement in Theorem 3:2. These theorems are applicable here since this result is independent of normality.

4. *The extremals and the canonical variables. An extremal for problem* A is defined to be an arc without corners in D_1 and a set of multipliers

$$y_i(t), \quad \chi_\alpha(t) \qquad (i = 1,\dots,n; \ \alpha = 0,1,\dots,m; \ t_1 \leqq t \leqq t_2) \tag{4:1}$$

for which the functions $y_i'(t)$, $\chi_\alpha(t)$ have continuous partial derivatives and satisfy the Euler-Lagrange equations (3:5). For an extremal, the first equation in (3:4) is a consequence of the equations (3:5) (IV, 32). An extremal is said to be *non-singular* if along it the matrix R in (3:6) has rank $n+m+1$. *An extremal for problem* B is defined to be an arc without corners in $\mathcal{R}_1$ and a unique set of multipliers

$$y_i(t), \quad \chi_\beta(t) \qquad (i = 1,\dots,n; \ \beta = 1,\dots,m; \ t_1 \leqq t \leqq t_2) \tag{4:2}$$

for which the functions $y_i'(t)$, $\chi_\beta(t)$ have continuous derivatives and satisfy the Euler-Lagrange equations (3:9). It is said to be *non-singular* if along it the matrix $\mathcal{R}$ in (3:10) has rank $n+m-1$.

In Section 8 of his lectures on the problen of Bolza (IV), Bliss gives a method of imbedding a non-singular extremal arc E for problem A in a 2n-parameter family of such extremals by introducing the canonical variables (t,y,z) related to the variables (t,y,y',λ) by the equations

$$(4:3) \qquad z_i = F_{y_i'}(t,y,y',\lambda), \quad 0 = \phi_\alpha(t,y,y') \quad (i=1,\dots,n;\ \alpha=0,1,\dots,m).$$

Along E these equations define functions $z_i(t)$ with continuous derivatives on t_1t_2 and the functional determinant of the right members with respect to y', λ is the determinant of the matrix R in (3:6). By the usual implicit function theorems there exists a neighborhood N of the solutions (t,y,z) belonging to E in which the equations (4:3) have solutions

$$(4:4) \qquad y_i' = P_i(t,y,z), \quad \lambda_\alpha = L_\alpha(t,y,z) \quad (i=1,\dots,n;\ \alpha=0,1,\dots,m)$$

reducing to $y_i'(t)$, $\lambda_\alpha(t)$ at each point [t, y(t), z(t)] belonging to E, and which are such that P_i, L_α have continuous derivatives of at least the second order. In the neighborhood N every solution of the equations

$$(4:5) \qquad y_i' = P_i(t,y,z), \quad z_i' = F_{y_i}[t,\ y,\ P(t,y,z),\ L(t,y,z)]$$

defines by means of equations (4:4) functions $L_\alpha(t)$ with which it satisfies equations (4:3) and hence the equations (3:5).

In the neighborhood N the Hamilton function H(t,y,z) may be defined by the equation

$$(4:6) \qquad H(t,y,z) = z_iP_i(t,y,z) - F(t,y,P,L)$$

which is the function into which the expression $y_i'F_{y_i'} - F$ is transformod by the substitution of the values of y_i' and λ_α from equations

(4:4). From equations (4:3) and simple calculations it follows that the partial derivatives of the function H have the values

(4:7) $$H_t = -F_t, \qquad H_{y_i} = -F_{y_i}, \qquad H_{z_i} = P_i.$$

Hence, equations (4:5) are equivalent to the system

(4:8) $$y_i' = H_{z_i}, \qquad z_i' = -H_{y_i}.$$

These are called the *canonical equations of the extremals*. Some of the consequences of the above theory are summarized in the following theorem, proved by Bliss (IV, 36).

THEOREM 4:1. *The imbedding theorem for problem* A. *Every non-singular extremal* E *for problem* A *is embedded for values* $t_1 \leqq t \leqq t_2$, t_0, $a_{10}, \dots, a_{n0}$, $b_{10}, \dots, b_{n0}$ *in a* 2n-*parameter family of extremals defined by functions of the form*

(4:9) $$y_i(t, t_0, a, b), \qquad z_i(t, t_0, a, b)$$

the functions y_i, y_{it}, z_i, z_{it} *have continuous partial derivatives of at least the second order in a neighborhood of the values* (t,a,b) *defining* E, *and the matrix*

$$\begin{Vmatrix} y_{ia_k} & y_{ib_k} \\ z_{ia_k} & z_{ib_k} \end{Vmatrix} \qquad (i,k = 1, \dots, n)$$

has rank 2n *along* E.

With the aid of equations (4:8), the following property of the function H is readily established.

THEOREM 4:2. *Along every solution* $y_i(t)$, $z_i(t)$ *of equations* (4:8),

(4:10) $$dH/dt = H_t.$$

For, if each i-th one of the first n equations in (4:8) is multiplied by z_i' and the second n by $-y_i'$ and if then the entire set is summed, the equation $0 = H_{y_i}y_i' + H_{z_i}z_i'$ is obtained, from which the equation (4:10) follows.

THEOREM 4:3. *Along every solution* $y_i(t)$, $z_i(t)$ *of equations* (4:8) *for problem* C, *the equation*

$$H(y,z) = L_0(y,z) \qquad (4:11)$$

holds identically; furthermore, $H(y,z)$ *has a constant value not identically zero.*

The relation (4:11) follows from condition (3:7). The last part of the theorem follows from the first equation in (3:4), as well as from Theorem 4:2, and the hypothesis that $y_i'y_i' \neq 0$.

THEOREM 4:4. *In the functions* (4:9) *for problem* C, *one of the parameters* b *can be taken equal to* H.

By Theorems 4:3 and 4:1, the equation $\chi_0 = H(a,b)$ is satisfied by the values (a,b,χ_0) belonging to E. Since not all of the functions H_{z_i} vanish at any point of E, there is a neighborhood of the values (a,b) defining E in which one of them, say H_{z_n}, remains different from zero. By implicit function theory the equation $\chi_0 = H(a,b)$ may be solved for b_n in the neighborhood just described, and

$$b_n = b_n(a_1, \ldots, a_n, b_1, \ldots, b_{n-1}, \chi_0). \qquad (4:12)$$

THEOREM 4:5. *The extremals for problem* B *satisfy the equations*

$$y_i' = H_{z_i}(y,z), \qquad z_i' = -H_{y_i}(y,z), \qquad H(y,z) = 0.$$

This theorem follows at once from Theorem 4:3 since the extremals for problem B are the projections in the y-space of those extremals for problem C for which $L_0(y,z)$ is equal to zero.

THEOREM 4:6. *The imbedding theorem for problem* B. *Every non-singular extremal* E *for problem* B *in* y-*space is imbedded for values* $t_{1o} \leq t \leq t_{2o}, A_{1o}, \dots, A_{n-1,o}, B_{1o}, \dots, B_{n-1,o}$ *in a* (2n - 2)-*parameter family of extremals defined by functions of the form*

$$Y_i(t,A,B), \qquad Z_i(t, A, B).$$

The functions Y_i, Y_{it}, Z_i, Z_{it} *have continuous partial derivatives of at least the second order in a neighborhood of the values* (t,A,B) *defining* E *and the matrix*

$$(4{:}13) \qquad \left\| \begin{matrix} Y_i' & Y_{iA_k} & Y_{iB_k} \\ Z_i' & Z_{iA_k} & Z_{iB_k} \end{matrix} \right\| \qquad (i = 1, \dots, n;\ k = 1,\dots,n-$$

has rank 2n - 1 *along* E.

By Theorem 4:4, the functions (4:9) for problem C may be written in the form

$$(4{:}14) \qquad \begin{aligned} y_i[t,a,B,b_n(a,B,H)] &= Y_i(t,a,B,H) \\ z_i[t,a,B,b_n(a,B,H)] &= Z_i(t,a,B,H), \end{aligned}$$

in which the symbols a, B represent the set $(a_1,\dots,a_n,b_1,\dots,b_{n-1})$. The projections in the y-space of the 2n-parameter family of solutions of equations (4:8) for problem C, for which $L_0(y,z) = 0$, form a (2n - 2)-parameter family of curves in the y-space. For, if an extremal projection in the y-space intersects a sub-space $a_i = \xi_i(A_1, \dots, A_{n-1})$, having continuous second partial derivatives in a neighborhood of the intersection and is not tangent to it, then every other projection near the extremal will also intersect without being tangent to it. Hence, the point of intersection of each extremal with the sub-space can be taken as the initial point ξ_i of the extremal for $t = t_{10}$. Since the only projections important for problem B are those for which $\chi_0 = 0$ and since $\chi_0 = L_0(y,z) = H(y,z)$ is a constant not identically zero along every extremal for problem C, the number of essential parameters is further reduced by one. The extremals for problem B are then defined by functions of the form

$$(4:15) \qquad Y_i(t,A,B), \qquad Z_i(t,A,B)$$

obtained from (4:14) by replacing a_i by $\xi_i(A_1, \ldots, A_{n-1})$ and setting $H = 0$.

Because of the identities

$$a_i = \xi_i(A) = Y_i(t_{10},A,B), \qquad B_j = Z_j(t_{10},A,B),$$

$$b_n[\xi(A),B] = Z_n(t_{10},A,B)$$

and the following relations deduced from them by differentiation with respect to the variables A_j and B_j,

$$\xi_{iA_j} = Y_{iA_j}, \qquad 0 = Y_{iB_j}, \qquad 0 = Z_{rA_j}, \qquad \delta_{rj} = Z_{rB_j},$$

$$b_{na_i}\xi_{iA_j} = Z_{nA_j}, \qquad b_{nB_j} = Z_{nB_j},$$

it is seen that at $t = t_{10}$ the matrix (4:13) is equal to the matrix

$$\left\| \begin{matrix} Y_i' & \xi_{iA_k} & 0 \\ Z_j' & 0 & \delta_{rk} \\ Z_n' & b_{na_i}\xi_{iA_k} & b_{nB_k} \end{matrix} \right\| \qquad \begin{matrix} (i = 1, \ldots, n) \\ (j,k,r = 1, \ldots, n-1) \\ (\delta_{rr} = 1, \ \delta_{rk} = 0 \text{ if } r \neq k) \end{matrix}$$

which has rank $2n - 1$ since the determinant $| Y_i' \quad \xi_{iA_k} |$ is different from zero at $t = t_{10}$. It follows that the matrix (4:13) must have rank $2n - 1$ along E. For, if the rank were less than $2n - 1$ at a point $t = t'$ on E there would exist constants c, c_k, d_k not all zero such that at that point

$$(4{:}16) \qquad cY_i' + c_kY_{iA_k} + d_kY_{iB_k} = 0, \quad cZ_i' + c_kZ_{iA_k} + d_kZ_{iB_k} = 0.$$

Moreover, since the functions in (4:15) satisfy the equations in Theorem 4:5 identically in t, A, B, the columns of the matrix (4:13) are solutions of the linear, homogeneous equations

$$\eta_i' = H_{z_iy_k}\eta_k + H_{z_iz_k}\zeta_k, \qquad \zeta_i' = -H_{y_iy_k}\eta_k - H_{y_iz_k}\zeta_k.$$

The functions

$$\eta_i(t) = cY_i' + c_kY_{iA_k} + d_kY_{iB_k}, \qquad \zeta_i(t) = cZ_i' + c_kZ_{iA_k} + d_kZ_{iB_k}$$

are also solutions of these equations and since $\eta_i(t_1) = \zeta_i(t_1) = 0$ and $\eta_i'(t_1) = \zeta_i'(t_1) = 0$, it follows that $\eta_i(t) \equiv 0 \equiv \zeta_i(t)$. Hence, the relations (4:16) would also have to hold at $t = t_0$ which is impossible since it has been proved that the matrix in (4:13) has rank $2n - 1$ at that point.

For the Problem of Mayer it will be assumed that the set of multipliers $\lambda_\alpha(t)$ associated with an extremal arc E do not vanish simultaneously. Since the matrix $\|\phi_{\alpha y_i'}\|$ has rank m, this assumption is equivalent to the assumption that the functions $z_i(t)$ associated with E do not vanish simultaneously. Hence, not all the constants b_{i0} in Theorem 4:1 are zero at the initial element on E. A similar remark applies for the problem of Mayer in parametric form.

Bliss and Hestenes (III, 309) have proved that for the problem of Mayer in non-parametric form the functions $P_i(t,y,z)$ and $L_\alpha(t,y,z)$ in (4:4) satisfy the homogeneity conditions

$$(4{:}17)\quad P_i(t,y,kz) = P_i(t,y,z), \quad L_\alpha(t,y,kz) = kL_\alpha(t,y,z) \qquad (k \neq 0).$$

These conditions imply the following theorem.

THEOREM 4:7. For the problem of Mayer in non-parametric form the Hamilton function H in (4:6) satisfies the condition

$$(4{:}18)\qquad H(t,y,kz) = kH(t,y,z) \qquad (k \neq 0),$$

and for proper choice of the parameters the functions in (4:9) satisfy the conditions

$$(4{:}19)\quad y_i(t,a,kb) = y_i(t,a,b), \quad z_i(t,a,kb) = kz_i(t,a,b) \qquad (k \neq 0).$$

The condition (4:18) follows from the first set of conditions in (4:17) and the fact that the function F in (3:1) for the problem of Mayer is identically zero along an arc satisfying the equations $\phi_\alpha = 0$. The last statement in the theorem (III, 310) is a consequence of the conditions (4:17) and

$$F(t,y,y',k\lambda) = kF(t,y,y',\lambda) \qquad (k \neq 0).$$

THEOREM 4:8. For the problem of Mayer in parametric form the Hamilton function H in (4:11) satisfies the condition

$$(4{:}20)\qquad H(y,kz) = kH(y,z) \qquad (k \neq 0),$$

and for proper choice of parameters the functions in (4:15) satisfy the relations

$$(4{:}21)\quad Y_i(t,A,kB) = Y_i(t,A,B), \quad Z_i(t,A,kB) = kZ_i(t,A,B) \qquad (k \neq 0).$$

From the condition (4:18) for the corresponding non-parametric problem, it follows that the condition (4:20) must hold and that the function b_n in (4:12) satisfies the homogeneity condition $b_n(a,kb_1, \ldots, kb_{n-1},k\chi_0) = kb_n(a,b_1, \ldots, b_{n-1},\chi_0)$. By virtue of the conditions in (4:19), the functions in (4:14) must satisfy the homogeneity relations

$$\begin{aligned} y_i[t,a,kB,b_n(a,kB,k\chi_0)] &= y_i[t,a,B,b_n(a,B,\chi_0)], \\ z_i[t,a,kB,b_n(a,kB,k\chi_0)] &= kz_i[t,a,B,b_n(a,B,\chi_0)]. \end{aligned} \qquad (k \neq 0)$$

Since these relations are not affected by replacing a_i by $\xi_i(A)$, the conditions given in (4:21) must hold for this choice of parameters.

5. *The necessary conditions of Weierstrass and Clebsch.* In this section the necessary condition of Weierstrass is given, both in the ordinary and the canonical variables, for problems A and B, as well as the special forms this condition assumes for the corresponding problems of Mayer. Further properties of the Hamilton function are developed and, finally, the condition of Clebsch for the problems A and B is given both in terms of the ordinary variables and the canonical variables.

THEOREM 5:1. *The necessary condition of Weierstrass for problem* A. *At each element* t,y,y',χ *of a normal minimizing extremal arc* E *for problem* A, *the inequality*

$$(5:1)\quad \mathcal{E}(t,y,y',\chi,Y') = F(t,y,Y',\chi) - F(t,y,y',\chi) - (Y_i' - y_i')F_{y_i'}(t,y,y',\chi) \geq 0$$

must hold for all sets $(t,y,Y') \neq (t,y,y')$ *interior to the region* D_1 *and satisfying the equations* $\phi_\alpha = 0$.

This theorem has been proved by Bliss using a method of Graves (IV, 58).

THEOREM 5:2. *The necessary condition of Weierstrass for problem* B. *At each element* y, y', λ *of a normal minimizing extremal for problem* B, *the inequality*

$$\mathcal{E}(y,y',\lambda,Y') = F(y,Y',\lambda) - F(y,y',\lambda) - (Y_i' - y_i')F_{y_i'}(y,y',\lambda) \geqq 0$$

must hold for all sets $(y,Y') \neq (y,y')$ *interior to the region* $\mathcal{R}_1$ *and satisfying the equations* $\phi_\beta = 0$.

This theorem is a consequence of Theorem 5:1 for problem C. Since, along a minimizing extremal arc E' for problem C, the multiplier λ_0 must be equal to zero by Theorem 3:2, the $\mathcal{E}$-function (5:1) for this arc is independent of ϕ. Hence, its projection E in the y-space, which is a minimizing extremal for problem B, must satisfy this theorem.

THEOREM 5:3. *The necessary condition of Weierstrass for problem* A *in canonical variables. At each element* t,y,z *of a normal, non-singular, minimizing extremal arc* E *for problem* A, *the inequality*

$$\mathcal{E}(t,y,z,Y') = F(t,y,z,Y') + H(t,y,z) - Y_i'z_i \geqq 0 \tag{5:2}$$

must hold for all sets $(t,y,Y') \neq (t,y,H_z)$ *interior to the region* D_1 *and satisfying the equations* $\phi_\alpha = 0$.

This theorem is an immediate consequence of Theorem 5:1 and conditions (4:6), (4:7) and (4:8).

THEOREM 5:4. *The necessary condition of Weierstrass for problem* B *in canonical variables. At each element* y, z *of a normal, non-singular minimizing extremal arc* E *for problem* B *satisfying the equation* $H(y,z) = 0$, *the inequality*

(5:3) $$\mathcal{E}(y,z,Y') = F(y,z,Y') - Y_i'z_i \geqq 0$$

must hold for all sets $(y,Y') \neq (y,H_z)$ *interior to the region* $\mathcal{R}$*; and satisfying the equations* $\phi_\beta = 0$.

This theorem follows at once from Theorem 5:3 for problem C with the aid of Theorems 3:2, 3:4 and 4:3.

Along an admissible arc for *the problem of Mayer in non-parametric form* satisfying the equations $\phi_\alpha = 0$, the function F in (3:1) is identically zero. From the homogeneity relation (4:18), it is found by differentiating with respect to k and setting k = 1 that

(5:4) $$z_iH_{z_i} = H.$$

For this problem,condition (5:2) becomes

$$\mathcal{E}(t,y,z,Y') = z_i(H_{z_i} - Y_i') \geqq 0.$$

Similarly, it can be shown that condition (5:3) for the *problem of Mayer in parametric form* becomes

$$\mathcal{E}(y,z,Y') = -Y_i'z_i \geqq 0.$$

THEOREM 5:5. *The necessary condition of Clebsch for problem* A. *At each element* t, y, y', λ *of a normal minimizing extremal arc* E *for problem* A, *the inequality*

(5:5) $$F_{y_i'y_k'}(t,y,y',\lambda)\pi_i\pi_k \geqq 0 \qquad (i,k = 1, \ldots, n)$$

must be satisfied for every set $(\pi_1, \ldots, \pi_n) \neq (0, \ldots, 0)$ *satisfying the equations*

(5:6) $$\phi_{\alpha y_i'}(t,y,y')\pi_i = 0 \qquad (\alpha = 0, 1, \ldots, m).$$

If E *is non-singular, the equality sign in* (5:5) *holds only in case*

$(\pi) = (0)$.

The first part of the theorem has been proved by Bliss (IV, 65). If the equality in (5:5) holds for some value $(\pi) \neq (0)$ satisfying the equations (5:6), then from the theory of minima for functions of n variables (IV, 40), it follows that there exist multipliers λ_α such that

$$\begin{aligned} F_{y_i' y_k'}\pi_k + \lambda_\alpha \phi_{\alpha y_i'} &= 0 \\ \phi_{\alpha y_k'}\pi_k &= 0. \end{aligned}$$

This is possible only in case the determinant of the matrix R in (3:6) is zero.

THEOREM 5:6. *The necessary condition of Clebsch for problem* B. *At each element* y, y', χ *of a normal minimizing extremal arc* E *for problem* B, *the inequality*

$$(5:7) \qquad F_{y_i' y_k'}(y, y', \chi)\pi_i \pi_k \geq 0 \qquad (i,k = 1, \ldots, n)$$

must be satisfied for every set $(\pi_1, \ldots, \pi_n) \neq (0, \ldots, 0)$ *satisfying the equations*

$$(5:8) \qquad \phi_{\beta y_i'}(y, y')\pi_i = 0 \qquad (\beta = 1, \ldots, m).$$

If E *is non-singular, the equality sign in* (5:7) *holds only in case* $(\pi) = (py')$.

This theorem is a consequence of Theorem 5:5 for problem C which states that for every set π_i satisfying the equations (5:8) and also $\phi_{oy_i'}(y, y')\pi_i = 0$, the condition (5:7) must be satisfied. But to every set π_i satisfying the equations (5:8) there corresponds a set $\pi_i - py_i'$, with $p = \phi_{y_j'}\pi_j$, satisfying the equations (5:8) and also the equation $\phi_{oy_i'}(\pi_i - py_i') = 0$ on account of the homogeneity of the functions ϕ_β and ϕ. For this set $\pi_i - py_i'$,

$$F_{y_i'y_k'}\pi_i\pi_k = F_{y_i'y_k'}(\pi_i - py_i')(\pi_k - py_k')$$

because of the homogeneity of F. The last expression is $\geqq 0$ on account of the Clebsch condition for problem C. If E is non-singular, its corresponding arc E' for problem C has the same property, as shown in Section 3. Along such an arc E', by Theorem 5:5, the above expression is zero only in case $(\pi - py') = (0)$.

In the following lemma and its proof certain properties of the Hamilton function H will be deduced which will be useful in stating and establishing the Clebsch condition in canonical variables.

LEMMA 5:1. Along a non-singular extremal arc E for problem A, the matrix

$$(5{:}9) \qquad \mathcal{L} = \left\| \begin{matrix} H_{z_j z_k} & L_{\nu z_j} \\ L_{\beta z_k} & 0 \end{matrix} \right\| \qquad \begin{matrix} (\nu, \beta = 0, 1, \dots, m) \\ (j, k = 1, \dots, m) \end{matrix}$$

has rank $n+m+1$. Furthermore, the matrix $\|L_{\nu z_j}\|$ has rank $m+1$ and the matrix $\|H_{z_j z_k}\|$ has rank $n-m-1$ along E.

Since, by hypothesis, the matrix R in (3:6) has rank $n+m+1$, the linear equations

$$(5{:}10) \qquad \begin{aligned} e_i &= F_{y_i'y_j'}\pi_j + \phi_{\nu y_i'}\mu_\nu && (\nu, \alpha = 0, 1, \dots, m) \\ 0 &= \phi_{\alpha y_j'}\pi_j && (i, j = 1, \dots, n), \end{aligned}$$

can be solved for π_j and μ_ν for arbitrary values of $e_1, \dots, e_n$. In order to state the solutions explicitly, it is observed that if the variables $y_i' = H_{z_i}(t,y,z)$, $\lambda_\alpha = L_\alpha(t,y,z)$ are substituted in equations (4:3) and the resulting identities are differentiated with respect to z_k, then

$$(5{:}11)\qquad \phi_{\alpha y_j'} H_{z_j z_k} = 0, \qquad F_{y_i' y_j'} H_{z_j z_k} + \phi_{\nu y_i'} L_{\nu z_k} = \delta_{ik}$$

where $\delta_{ii} = 1$ and $\delta_{ik} = 0$ if $i \neq k$. Hence,

$$(5{:}12)\qquad \pi_j = H_{z_i z_j} e_i, \qquad \mu_\nu = L_{\nu z_i} e_i$$

are the solutions of equations (5:10).

If $e_i = \phi_{\alpha y_i'}$, then $\pi_j = 0$ by equations (5:12) and (5:11), and it follows from the first group of equations in (5:10) that

$$(5{:}13)\qquad L_{\nu z_k} \phi_{\alpha y_k'} = \delta_{\nu\alpha} \qquad (\alpha, \nu = 0, 1, \ldots, m).$$

By means of these equations and equations (5:11), it is found that the product of the determinant of the matrix R in (3:6) and that of the matrix $\mathcal{L}$ in (5:9) is

$$\begin{vmatrix} \delta_{ik} & F_{y_i' y_j'} L_{\nu z_j} \\ 0 & \delta_{\alpha\nu} \end{vmatrix} = 1 \qquad \begin{matrix} (i, k, j = 1, \ldots, n) \\ (\alpha, \nu = 0, 1, \ldots, m). \end{matrix}$$

Hence, the matrix $\mathcal{L}$ has rank $n + m + 1$. It follows at once that the matrix $\| L_{\nu z_j} \|$ must have rank $m + 1$ since otherwise the matrix $\mathcal{L}$ could not have rank $n + m + 1$. Similarly, the rank of the matrix $\| H_{z_j z_k} \|$ must be $\geqq n - m - 1$. But the first set of equations in (5:11), which represent $m + 1$ linearly independent sets of relations among the rows of this matrix, show that its rank must be $\leqq n - m - 1$. Hence, the rank of the matrix $\| H_{z_j z_k} \|$ is exactly $n - m - 1$.

THEOREM 5:7. *The necessary condition of Clebsch for problem* A *in canonical variables. At each element* (t,y,z) *of a normal non-singular minimizing extremal arc* E *for problem* A, *the inequality*

$$H_{z_i z_k} e_i e_k > 0 \qquad (i, k = 1, \ldots, n)$$

must be satisfied for every set $(e_1, \ldots, e_n) \neq (0, \ldots, 0)$ *satisfying the equations*

$$(5:14) \qquad L_{\nu z_i} e_i = 0 \qquad (\nu = 0, 1, \ldots, m).$$

Let e_i be a solution of equations (5:14). If π_i are defined by the expressions $H_{z_i z_k} e_k$, it follows from the first group of equations in (5:11) that π_i satisfy the equations $\phi_{\alpha y_i'} \pi_i = 0$. Conversely, let π_i be a solution of the equations $\phi_{\alpha y_i'} \pi_i = 0$. Then if e_i are defined by the expressions $F_{y_i' y_j'} \pi_j$, it is seen with the aid of the equations (5:10) and their solutions (5:12) that e_i must satisfy the equations (5:14). Furthermore, the numbers π_i are all zero if and only if the corresponding numbers e_i are all zero, since the determinant of the coefficients in the equations (5:10) is different from zero along a non-singular arc. It should be noted that equations (5:12) and (5:11) show that $e_i \neq h_\alpha \phi_{\alpha y_i'}$ since otherwise $(\pi) = (0)$. This condition on the sets (e) is implied by the equations (5:14) on account of the conditions (5:13). It is now seen with the aid of equations (5:10) and their solutions (5:12), that for every set $(\pi) \neq (0)$ satisfying the equations $\phi_{\alpha y_i'} \pi_i = 0$, there corresponds a set $(e) \neq (0)$ satisfying the equations (5:14), and conversely, for which

$$H_{z_i z_k} e_i e_k = F_{y_i' y_k'} \pi_i \pi_k .$$

The present theorem now follows from Theorem 5:5.

THEOREM 5:8. *The necessary condition of Clebsch for problem* B *in canonical variables. At each element* (y, z) *of a normal non-singular minimizing extremal arc* E *for problem* B, *the inequality*

$$(5:15) \qquad H_{z_i z_k} e_i e_k > 0 \qquad (i, k = 1, \ldots, n)$$

must be satisfied for every set $(e_1, \ldots, e_n) \neq (p\phi_{y_1'}, \ldots, p\phi_{y_n'})$ *satisfying the equations*

$$(5:16) \qquad L_{\beta z_i} e_i = 0, \qquad (\beta = 1, \ldots, m)$$

where the function ϕ *is the function appearing in the function* ϕ_o *defined by equation* (1:5).

This theorem is a consequence of Theorem 5:7 for problem C which states that for every set $(e_1, \ldots, e_n) \neq (0, \ldots, 0)$ satisfying equations (5:16) and also $L_{oz_i}e_i = 0$, the condition (5:15) must be satisfied. But to every set e_i satisfying the equations (5:16) there corresponds a set $e_i - p\phi_{y_i'}$ with $p = L_{oz_k}e_k$, satisfying the conditions (5:16) and also the equation $L_{oz_i}(e_i - p\phi_{oy_i'}) = 0$ on account of the condition (5:13). For this set $e_i - p\phi_{oy_i'}$

$$H_{z_i z_k} e_i e_k = H_{z_i z_k}(e_i - p\phi_{oy_i'})(e_k - p\phi_{oy_k'})$$

because of the first group of equations in (5:11). The right member of the last expression must be positive for $(e - p\phi_{oy'}) \neq (0)$ by Theorem 5:7 for problem C.

6. *Mayer fields for problem A*. A Mayer field, as defined by Bliss, is a region $\mathcal{F}'$ in ty-space containing only interior points and having associated with it a set of slope functions and multipliers $p_i(t,y)$, $\lambda_\alpha(t)$, having continuous first partial derivatives in $\mathcal{F}'$, defining elements $[t,y,p(t,y)]$ which lie in the region D_1 and satisfy the equations $\phi_\alpha = 0$ for every (t,y) in $\mathcal{F}'$, and making the integral

$$I^* = \int (F - p_i F_{y_i'})dt + F_{y_i'}\, dy_i$$

independent of the path in $\mathcal{F}'$. The arguments of F and its derivatives in the integrand of I^* are $[t,y,p(t,y),\chi(t,y)]$.

Bliss has shown that the solutions of the equations $y_i' = p_i(t,y)$ are extremals with multipliers $\chi_\alpha[t,y(t)]$, called *the extremals of the field* (IV, 103). It should be noted that for the problem of Mayer the function $F(t,y,p,\chi)$ vanishes identically in $\mathcal{F}'$ and the integral I^* has the value zero along every extremal of the field.

Let $\bar{D}$ be a region of points (t,y,y',χ) with (t,y,y') interior to the region D_1 described in Section 2 and so related to a region N of points (t,y,q) that to each point (t,y,q) interior to N there corresponds a unique solution (t,y,y',χ) of equations (4:3) with (t,y,y',χ) interior to $\bar{D}$ and the determinant of the matrix R in (3:6) different from zero. The equations (4:3) with $z_i = q_i$ have as before single-valued solutions with continuous second derivatives over the interior of N, and every solution $y_i(t)$, $z_i(t) = q_i(t)$ of equations (4:8) in this region defines an extremal. A set (t,y,q) interior to N will be called *an admissible set*.

A second definition of a Mayer field is a region $\mathcal{F}$ of ty-space containing only interior points with which are associated functions $q_i(t,y)$ having continuous first partial derivatives in $\mathcal{F}$, defining elements $[t,y,q(t,y)]$ interior to the region N for every (t,y) in $\mathcal{F}$, and making the integral

$$I^* = \int q_i(t,y)dy_i - H[t,y,q(t,y)]dt$$

independent of the path in $\mathcal{F}$.

THEOREM 6:1. *If the integral I^* is independent of the path in $\mathcal{F}$, the equations*

$$(6:1) \qquad -\partial H/\partial y_i = \partial q_i/\partial t, \qquad \partial q_i/\partial y_k = \partial q_k/\partial y_i \quad (i,k = 1, \ldots, n)$$

hold identically in the variables (t,y).

From the hypothesis of the theorem, it follows that every arc in $\mathcal{F}$ is a minimizing arc for the integral I^* and hence must satisfy the corresponding Euler-Lagrange differential equations for I^*. In order that this be true, equations (6:1) must hold.

Along an admissible arc $y_i = y_i(t)$ in $\mathcal{F}$, the identity

$$(6:2) \qquad \partial q_i/\partial t + \partial H/\partial y_i = dq_i/dt + H_{y_i} + q_{iy_k}(H_{z_k} - y_k')$$

is readily seen to be true since

$$\partial H/\partial y_i = H_{y_i} + H_{z_k}q_{ky_i} = H_{y_i} + H_{z_k}q_{iy_k}, \qquad dq_i/dt = q_{it} + q_{iy_k}y_k'.$$

THEOREM 6:2. Every solution $y_i(t)$ in the field $\mathcal{F}$ of the equations

$$(6:3) \qquad dy_i/dt = H_{z_i}[t,y,q(t,y)]$$

is an extremal with $z_i(t) = q_i[t,y(t)]$.

The sets $[t,y,q(t,y)]$ are interior to N for (t,y) in $\mathcal{F}$, and equations (6:1) and (6:2) show that along an arc satisfying equations (6:3), the equation

$$dq_i/dt + H_{y_i} = 0$$

must hold. If $z_i(t)$ is defined to be $q_i[t,y(t)]$, then $y_i(t)$ and $z_i(t)$ satisfy the equations

$$dy_i/dt = H_{z_i}(t,y,z), \qquad dz_i/dt = -H_{y_i}(t,y,z),$$

which are the canonical equations of the extremals. The arcs satisfying the equations (6:3) with $z_i(t) = q_i[t,y(t)]$ are called the extremals of the field.

THEOREM 6:3. Through each point of the field $\mathfrak{F}$ there passes one and only one extremal of the field.

This theorem follows from the fact that the equations (6:3) are of first order.

A Mayer field as defined by Bliss is a field according to the second definition, provided that the sets $[t,y,q(t,y)]$ defined by points (t,y) in $\mathfrak{F}'$, where $q_i(t,y) = F_{y_i'}[t,y,p(t,y),\chi(t,y)]$, are interior to the region N. Every field of the second type is a Mayer field as defined by Bliss with $p_i(t,y) = H_{z_i}[t,y,q(t,y)]$ and $\chi_\alpha(t,y) = L_\alpha\,[t,y,q(t,y)]$.

7. Transversal surfaces of a Mayer field for problem A. Let $\mathfrak{F}$ be a field defined by functions $q_i(t,y)$. An n-space $W(t,y) =$ constant is called a transversal surface of the field if at each point of the n-space the equation

$$q_i(t,y)dy_i - H[t,y,q(t,y)]dt = 0$$

is satisfied by every set of differentials dt, dy_i which satisfy the equation

$$W_{y_i}\, dy_i + W_t\, dt = 0.$$

It is evident that a necessary and sufficient condition that an n-spacc $W(t,y) =$ constant be a transversal surface of a field $\mathfrak{F}$ is that at each point on the n-space in $\mathfrak{F}$

$$W_{y_i} = kq_i, \qquad W_t = -kH \tag{7:1}$$

where k is a proportionality factor.

THEOREM 7:1. *In every field* $\mathcal{F}$ *the Hilbert Integral defines a function* $W(t,y)$ *having continuous second partial derivatives in* $\mathcal{F}$, *satisfying the Hamilton-Jacobi partial differential equation*

$$W_t + H(t,y,W_y) = 0, \tag{7:2}$$

and such that the surfaces of the family $W(t,y)$ = constant *are the transversal surfaces of the field. Conversely, let* $W(t,y)$ *be a function which has continuous second partial derivatives and defines only admissible elements* $(t,y,q) = (t,y,W_y)$ *in a region* $\mathcal{F}$ *of* ty-*space. Then if* W *satisfies equation* (7:2), *the region* $\mathcal{F}$ *is a field with*

$$q_i(t,y) = W_{y_i}, \tag{7:3}$$

and the surfaces $W(t,y)$ = constant *are the transversal surfaces of the field.*

In a field $\mathcal{F}$ determined by a set of functions $q_i(t,y)$, the function

$$W(t,y) = \int_{(t_0,y_0)}^{(t,y)} q_i(t,y)dy_i - H[t,y,q(t,y)]dt, \tag{7:4}$$

where (t_0,y_0) is a fixed point in $\mathcal{F}$, has the properties described in the first part of the theorem since its derivatives are given by equations (7:1) with $k = 1$. The equations (7:1) with $k = 1$ are clearly equivalent to equation (7:2).

Conversely, every solution W of equation (7:2) with the properties described in the theorem defines functions q_i by means of equations (7:3) which with W satisfy the equations $W_t = -H(t,y,q)$, $q_i = W_{y_i}$. The Hilbert integral I^* formed with the functions $q_i = W_{y_i}$ is the integral of dW and hence is independent of the path.

Thus the region $\mathfrak{F}$ is a field with functions $q_i = W_{y_i}$, according to the second definition in Section 6.

The following well-known properties of a field are easily established for a field $\mathfrak{F}$. In a field, $I^*(C_{12}) = W(t,y)\Big|_1^2$ for any two points 1 and 2. Therefore, for every arc C_{12} on a transversal surface, $I^*(C_{12})$ is identically zero. From this it follows that, if the extremals of the field do not lie on the transversal surfaces, then two transversal surfaces intercept arcs on the extremals of the field along which the integral $\int (z_i H_{z_i} - H)dt$ has a constant value.

8. *Construction of fields and relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem* A. In this section conditions are given under which it is possible to construct an n-parameter family of fields from a 2n-parameter family of extremals. The complete integral of the Hamilton-Jacobi equation is determined with the aid of a 2n-parameter family of extremals, and conditions which this integral must satisfy if some of the parameters do not appear in the functions y_i defining the extremals are derived. Finally it is shown how to determine a 2n-parameter family of extremals from a complete integral, and conditions are given which imply the non-appearance of some of the parameters in the functions y_i defining the extremals.

THEOREM 8:1. *Suppose that a non-singular extremal arc* E *for problem* A *is a member for values* $t_{10} \leqq t \leqq t_{20}$, $a_{10}, \ldots, a_{n0}$, $b_{10}, \ldots, b_{n0}$ *of a* 2n-*parameter family of extremals defined by functions of the form*

$$(8{:}1) \qquad y_i(t, a_1, \ldots, a_n, b_1, \ldots, b_n), \quad z_i(t, a_1, \ldots, a_n, b_1, \ldots, b_n)$$

which are such that the functions y_i, y_{it}, z_i, z_{it} *have continuous* *first derivatives in a neighborhood* $\mathcal{B}$ *of the values* (t,a,b) *belonging to* E. *Suppose further that for each set* (t,a,b) *in* $\mathcal{B}$ *the determinants*

$$(8{:}2) \qquad \left| y_{ia_k} \right| , \qquad \begin{vmatrix} y_{ia_k} & y_{ib_k} \\ z_{ia_k} & z_{ib_k} \end{vmatrix}$$

are different from zero and the equations

$$(8{:}3) \qquad y_{ia_k} z_{ia_j} - y_{ia_j} z_{ia_k} = 0 \qquad (i,k,j = 1, \ldots, n)$$

hold identically. Then for fixed values of (b) *in* $\mathcal{B}$ *the family of extremals defined by the functions* (8:1) *form a Mayer field over a neighborhood* $\mathcal{F}$ *of the points* (t,y) *on* E. *Furthermore, the functions* $q_i(t,y,b)$ *for each field have the determinant*

$$(8{:}4) \qquad \left| q_{ib_k} \right| \qquad (i,k = 1, \ldots, n)$$

different from zero. The functions $q_i(t,y,b)$ *are defined by the equations*

$$(8{:}5) \qquad q_i(t,y,b) = z_i[t,a(t,y,b),b],$$

and the functions $a_i(t,y,b)$ *are found by solving the equations*

$$(8{:}6) \qquad y_i = y_i(t,a,b).$$

Since the determinant $| y_{ia_k} |$ is different from zero for (t,a,b) in $\mathcal{B}$, the system of equations (8:6) has a unique solution $a_i = a_i(t,y,b)$ for (b) in $\mathcal{B}$ and (t,y) in a neighborhood $\mathcal{F}$ of the points (t,y) on E; the functions $a_i(t,y,b)$ have continuous second derivatives for (b) in $\mathcal{B}$ and (t,y) in $\mathcal{F}$ since the right members of equations (8:6) have such derivatives. It follows from conditions (8:3) that for each fixed set (b) in $\mathcal{B}$ the family of extremals defined by the functions (8:1) form a Mayer field over the region $\mathcal{F}$ with functions $q_i(t,y,b)$ defined as in (8:5) (I, 595). There is a family of such fields depending upon the n parameters (b).

In each field the determinant (8:4) must be different from zero. For, the differentials of the equations (8:5) and the identities $y_i = y_i[t,a(t,y,b),b]$ with respect to b_j satisfy the equations

$$q_{ib_j}db_j = z_{ia_k}da_k + z_{ib_j}db_j, \qquad (i,j,k = 1, \ldots, n)$$

$$0 = y_{ia_k}da_k + y_{ib_j}db_j,$$

where $da_k = a_{kb_j}db_j$. If the determinant (8:4) were equal to zero at a point (t,y) in $\mathcal{F}$ and (b) in $\mathcal{B}$, there would exist values db_j not all zero satisfying the equation $q_{ib_j}db_j = 0$. For such values of db_j, the second determinant in (8:2) would have the value zero which is contrary to hypothesis.

The following theorem (II, 268; VIII, 146) indicates conditions under which an extremal satisfying the hypotheses of the previous theorem exists.

THEOREM 8:2. *Let E be a normal non-singular extremal for problem A for which there exists a conjugate system* η_{ik}, ζ_{ik} *of solutions of the canonical accessory equations*

$$(8:7)\quad \eta_i' = H_{z_iy_k}\eta_k + H_{z_iz_k}\zeta_k, \qquad \zeta_i' = -H_{y_iy_k}\eta_k - H_{y_iz_k}\zeta_k$$

whose determinant $|\eta_{ik}(t)|$ *is different from zero along E. Then E is a member of a 2n-parameter family of extremals of the type described in Theorem* 8:1.

The equations (8:7) are derived by Bliss in his lectures on the Problem of Bolza (IV, 75). To establish the theorem let

$$
(8:8) \qquad \begin{aligned} y_i &= Y_i(t, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n), \\ z_i &= Z_i(t, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \end{aligned}
$$

be a 2n-parameter family of extremals containing E for values $\alpha_i = \alpha_{i0}$, $\beta_i = \beta_{i0}$, $t_{10} \leqq t \leqq t_{20}$ and such that the parameters α_i, β_i are the values of the functions Y_i, Z_i at $t = t_{10}$, the functions Y_i, Y_{it}, Z_i, Z_{it} having continuous first and second partial derivatives in a neighborhood of the values (t, α, β) belonging to E. The 2n-parameter family

$$
(8:9) \quad y_i = y_i(t, a_1, \ldots, a_n, b_1, \ldots, b_n), \quad z_i = z_i(t, a_1, \ldots, a_n, b_1, \ldots, b_n),
$$

obtained from the family (8:8) by substituting for (α, β) the functions $\alpha(a)$, $\beta(a,b)$ defined by the equations

$$
(8:10) \quad \alpha_i(a) = \alpha_{i0} + \eta_{ik}(t_{10}) a_k, \qquad \beta_i(a,b) = \beta_{i0} + \zeta_{ik}(t_{10}) a_k + b_i,
$$

contains E for $(a,b) = (0,0)$ and can be shown to have the properties described in Theorem 8:1. For, by differentiating the identities

$$
y_i(t_{10}, a, b) = \alpha_i(a), \qquad z_i(t_{10}, a, b) = \beta_i(a, b)
$$

with respect to a_k, b_k it is found that at $t = t_{10}$,

$$
(8:11) \quad y_{ia_k} = \eta_{ik}(t_{10}), \quad z_{ia_k} = \zeta_{ik}(t_{10}), \quad y_{ib_k} = 0, \quad z_{ib_k} = \delta_{ik}
$$

where $\delta_{ii} = 1$ and $\delta_{ik} = 0$ if $i \neq k$. Hence, along E

$$
y_{ia_k} = \eta_{ik}(t), \qquad z_{ia_k} = \zeta_{ik}(t)
$$

since along E the right and left members of these equations are solutions of equations (8:7) with the same initial values at $t = t_{10}$. The determinant $| y_{ia_k} |$ is therefore different from zero along E. It follows that the first n equations in (8:9) have unique solutions

$a_i = a_i(t,y,b)$ which are continuous and have continuous first and second derivatives for all (t,y,b) such that (t,y) is in a neighborhood $\mathcal{F}$ of points (t,y) on E and (b) in a neighborhood $\mathcal{B}$ of $(b) = (0)$. For each fixed set (b) in $\mathcal{B}$ and (a) in a neighborhood of (0),

$$y_{ia_k} z_{ia_j} - y_{ia_j} z_{ia_k} = \eta_{ik}(t_{io})\, \zeta_{ij}(t_{io}) - \eta_{ij}(t_{io})\, \zeta_{ik}(t_{io})$$

on the hyperplane $t = t_{io}$. The right members of these equations have the value zero since η_{ik}, ζ_{ik} is a conjugate system of solutions of the equations (8:7) (IV, 80). Since the left members of these equations are constant along each extremal of the family (8:9) for (t,a,b) in a neighborhood of the set (t,a,b) on E (I, 595), it follows that condition (8:3) of Theorem 8:1 is satisfied. With the aid of equations (8:11) it is seen that the second determinant in (8:2) is different from zero at $t = t_{io}$ on E and hence must be different from zero along E (IV, 37). Therefore, there is at least a neighborhood of the values (t,a,b) belonging to E in which the 2n-parameter family of extremals (8:9) satisfies the hypotheses of Theorem 8:1.

A complete integral of the Hamilton-Jacobi partial differential equation will now be determined with the aid of a 2n-parameter family of extremals.

THEOREM 8:3. *If a non-singular extremal arc E for problem A is a member of a 2n-parameter family of extremals having the properties described in Theorem 8:1, then there is a neighborhood of E in which, with an arbitrary additive constant, the function*

$$W(t,y,b) = \int_{(t_o,y_o)}^{(t,y)} q_i(t,y,b)\,dy_i - H(t,y,q)\,dt$$

with q_i *defined by equations* (8:5), *is a complete integral of the*

Hamilton-Jacobi partial differential equation $W_t + H(t,y,W_y) = 0$ *and has the determinant* $|W_{y_i b_k}|$ $(i,k = 1, \dots, n)$ *different from zero.*

This theorem follows directly from Theorem 8:1 and the first part of Theorem 7:1.

The theorem of Jacobi on the determination of the extremals from a complete integral of the Hamilton-Jacobi equation will now be given.

THEOREM 8:4. *If* $W(t,y_1, \dots, y_n,b_1, \dots, b_n)$ *has continuous second partial derivatives with the determinant*

$$\left| W_{b_i y_k} \right| \qquad (i,k = 1, \dots, n) \tag{8:12}$$

different from zero for all points (t,y,b) *satisfying conditions of the form* (t,y) *in a region* $\mathfrak{F}$, (b) *in a region* $\mathfrak{B}$, *and the sets* $(t,y,q) = (t,y,W_y)$ *corresponding to these arguments admissible, and if furthermore* W *satisfies the Hamilton-Jacobi partial differential equation*

$$W_t + H(t,y,W_y) = 0 \tag{8:13}$$

identically in (t,y,b), *then the equations*

$$W_{b_i}(t,y,b) = W_{b_i}(t_0,a,b) \tag{8:14}$$

define a 2n-*parameter family of extremals*

$$y_i(t,t_0,a,b), \qquad z_i(t,t_0,a,b)$$

with properties similar to those described in Theorem 4:1, *the functions* $z_i(t,t_0,a,b)$ *being defined by the equations*

$$z_i = W_{y_i}[t,y(t,t_0,a,b),b]. \tag{8:15}$$

Since the function W satisfies the equation (8:13) identically in (t,y,b), it follows that W must satisfy the equations

$$(8{:}16) \qquad W_{tb_i} + H_{z_k}W_{y_kb_i} = 0, \qquad W_{ty_i} + H_{y_i} + H_{z_k}W_{y_ky_i} = 0,$$

found by differentiating the equations (8:14) with respect to b_i and y_i, respectively.

It will now be shown that every admissible arc E interior to $\mathcal{F}$, having no corners, and satisfying the equations (8:14) for particular constant values a_{io}, b_{io} is an extremal. For, the functions $y_i(t)$ $(t_1 \leqq t \leqq t_2)$ defining E with the functions $z_i(t)$ defined by means of (8:15) satisfy the equations

$$(8{:}17) \qquad W_{b_it} + W_{b_iy_k}y_k' = 0, \qquad z_i' = W_{y_it} + W_{y_iy_k}y_k',$$

found by differentiating the equations (8:14) and (8:15) with respect to t. The equations (8:16) and (8:17) show, with the aid of the condition (8:12), that the functions $y_i(t)$, $z_i(y)$ satisfy the canonical equations (4:8) and hence that E is an extremal.

The values (t,y,t_o,a,b) belonging to E form a set satisfying equations (8:14) on which the determinant (8:12) is different from zero. Hence, equations (8:14) have solutions $y_i = y_i(t,t_o,a,b)$ which, substituted in (8:15), define functions $z_i = z_i(t,t_o,a,b)$. For a fixed value of t_o, these functions define a 2n-parameter family of extremals

$$(8{:}18) \qquad y_i(t,a,b), \qquad z_i(t,a,b)$$

including E for the parameter values $a_i = a_{io}$, $b_i = b_{io}$ and having continuous partial derivatives of at least the first order in a neighborhood of these values since the left members of equations (8:14) and the right members of equations (8:15) have such derivatives. Since, from the discussion above, the functions in (8:18) satisfy the canonical equations (4:8), it follows that y_{it}, z_{it} also have continuous first derivatives. By substituting the functions (8:18) in equations (8:14) and (8:15) and differentiating the resulting identities with respect to a_j, b_j, it is readily seen that the product of the determinants

$$\begin{vmatrix} W_{b_i y_k} & 0 \\ W_{y_i y_k} & -\delta_{ik} \end{vmatrix} \begin{vmatrix} y_{ka_j} & y_{kb_j} \\ z_{ka_j} & z_{kb_j} \end{vmatrix} \qquad (i,j,k = 1, \ldots, n)$$

is the determinant

$$\begin{vmatrix} W_{b_i y_j}(t_0,a,b) & W_{b_i b_j}(t_0,a,b) - W_{b_i b_j}(t,y,b) \\ 0 & -W_{y_i b_j}(t,y,b) \end{vmatrix}$$

and hence that the value of the functional determinant of (y,z) with respect to (a,b) has rank 2n. This establishes the final property of the family (4:9) as belonging also to the family (8:18).

For certain problems of Bolza some of the parameters (b) do not appear in the first n functions (8:1) defining a 2n-parameter family of extremals for the problem. For the problem of Mayer, for example, at least one of the parameters (b) is non-essential in the first n functions in (8:1) as shown by the homogeneity relations (4:19). A similar condition holds in the case of the problem of Lagrange with finite side conditions transformed by differentiation of the side conditions into a problem of Lagrange with differential equations as side conditions (VI, 11). It should be noted that in the first of these problems the order of abnormality (IV, 52) is at least 1 and in the second it is equal to $m+1$. In the following theorem a condition is given which is necessary and sufficient for the non-appearance of some of the parameters (b) in the first n functions in (8:1).

THEOREM 8:5. *If the last s of the parameters* (b) *do not appear in the functions* $y_i(t,a,b)$ *in* (8:1) *of Theorem* 8:1, *where* s *is one of the numbers* 1, ..., m + 1, *then the function* W *described in Theorem* 8:3 *satisfies the additional equations*

$$(8{:}19) \qquad H_{z_k z_i} W_{y_i b_\rho} = 0 \quad (\rho = n-s+1, \ldots, n;\ i,k = 1, \ldots, n).$$

Conversely, if the function W(t,y,b) *having the properties described in Theorem* 8:4 *has continuous third derivatives and also satisfies the equations* (8:19), *then the last* s *of the parameters* (b) *do not appear in the functions* $y_i(t,a,b)$ *in* (8:18) *defining the extremals.*

To establish the first statement in the theorem it should be noted that if the last s of the parameters $b_1, \ldots, b_n$ do not appear in the functions $y_i(t,a,b)$ in (8:1), then the functions $a_i(t,y,b)$ described in Theorem 8:1 are independent of these parameters also. Differentiating the identity

$$W_{y_i}(t,y,b) = z_i[t,a(t,y,b),b]$$

for b_ρ, it is found that $W_{y_i b_\rho} = z_{ib_\rho}$. Moreover, from the relation

$$y_i'(t,a,b) = H_{z_i}[t,y(t,a,b),z(t,a,b)]$$

one finds by differentiating for b_ρ that $H_{z_i z_k} z_{kb_\rho} = 0$ since the functions $y_i(t,a,b)$, $y_i'(t,a,b)$ are independent of b_ρ. Equations (8:19) follow easily from these relations.

To prove the last part of the theorem let W(t,y,b) be a solution of the Hamilton-Jacobi equation (8:13) as described in Theorem 8:4. By the latter part of Theorem 7:1, the region $\mathfrak{F}$ in ty-space for each fixed set (b) in $\mathcal{B}$ is a field with functions $q_i(t,y,b) = W_{y_i}(t,y,b)$ and with extremals of each field defined by the functions (8:18) in which the parameters a_i are assumed to be the values of the functions y_i at $t = t_0$. The invariant integral (7:4) from a fixed point (t_0,a) to a variable point (t,y) in $\mathcal{F}$ is given by

$$\int_{(t_0,a)}^{(t,y)} (q_k\, dy_k - H\, dt) = W(t,y,b) - W(t_0,a,b).$$

It follows that

$$(8:20) \qquad W_{b_i}(t,y,b) = W_{b_i}(t_0,a,b) + \int_{(t_0,a)}^{(t,y)} q_{kb_i}(dy_k - H_{z_k}dt),$$

$$(8:21) \qquad \begin{aligned} W_{b_ib_j}(t,y,b) = W_{b_ib_j}(t_0,a,b) + \int_{(t_0,a)}^{(t,y)} [q_{kb_ib_j}(dy_k - H_{z_k}dt) \\ - H_{z_kz_h} q_{kb_i} q_{hb_j}\, dt] \qquad (i,j,k,h = 1, \ldots, n). \end{aligned}$$

If the functions $y_i(t,a,b)$ are substituted for y_i in the equations (8:14) and the resulting identities are differentiated for b_ρ, it is found with the aid of the identities obtained by making the same substitution in equations (8:21), (8:19) and $W_{y_ib_\rho} = q_{ib_\rho}$ that

$$W_{b_iy_j} y_{jb_\rho} = - \int_{(t_0,a)}^{(t,y)} q_{kb_ib_\rho}(dy_k - H_{z_k}dt).$$

In view of the fact that the functions $y_i(t,a,b)$ define an extremal of the field passing through the point (t_0,a), the invariant line integral above may be evaluated along this extremal. When this is done, it is found to have the value zero. Accordingly, one obtains the relations $W_{b_iy_j} y_{jb_\rho} = 0$. Since the determinant in (8:12) is different from zero for (t,y) in $\mathcal{F}$ and (b) in $\mathcal{B}$, it follows that the functions $y_i(t,a,b)$ in (8:18) must, therefore, have the form stated in the theorem.

9. Fields for the problem of Mayer in non-parametric form. For this problem, as stated in Theorem 4:7, the Hamilton function $H(t,y,z)$ is homogeneous of order one in z. Because of this fact, the fields for this problem have certain peculiar properties which will be discussed in this section. It will be recalled also that it was shown in Section 4 that the multipliers $\chi_\alpha(t)$ associated with an extremal arc E for this problem vanish simultaneously if and only if the functions $z_i(t)$ associated with E have this property.

THEOREM 9:1. The integral I^* has the value zero along every extremal of a field $\mathfrak{F}$. Furthermore, every extremal of a field lies on one of its transversal surfaces $W(t,y)$ = constant and the values W_{y_i} are not all zero.

Along each extremal of a field, the integrand of the invariant integral I^* is equal to the expression $z_i H_{z_i} - H$ which is identically zero on account of the condition (5:4). It follows that each extremal of the field lies on the transversal surface $W(t,y) = W(t_0,y_0)$ defined by the function (7:4) where (t_0,y_0) is a fixed point on the extremal. The values W_{y_i} are not all zero since they are the values of the functions z_i belonging to the extremals of the field.

THEOREM 9:2. Let E be a non-singular extremal arc for the problem of Mayer having the properties described in Theorem 8:1. Suppose further that the functions $y_i(t,a,b)$, $z_i(t,a,b)$ in (8:1) satisfy the homogeneity conditions (4:19). Then the functions $q_i(t,y,b)$ defined in (8:5) satisfy the conditions

$$q_i(t,y,kb) = kq_i(t,y,b) \qquad (k \neq 0). \tag{9:1}$$

This theorem follows from the fact that the functions $a_i(t,y,b)$ described in Theorem 8:1 would, under the hypotheses of this theorem, satisfy the conditions $a_i(t,y,kb) = a_i(t,y,b)$ $(k \neq 0)$.

THEOREM 9:3. *Let E be a non-singular extremal for the problem of Mayer having the properties described in Theorem* 8:2. *Then E is a member of a 2n-parameter family of extremals of the type described in Theorem* 9:2.

Since the parameters α_i, β_i in the functions $Y_i(t,\alpha,\beta)$, $Z_i(t,\alpha,\beta)$ in (8:8) are the initial values of Y_i, Z_i at $t = t_0$ these functions satisfy the relations (4:19) and not all the parameters β_{io} belonging to E are zero (III, 310). In order to preserve this property for the functions $y_i(t,a,b)$, $z_i(t,a,b)$ in (8:9), the expressions for β_i in (8:10) are replaced by the following expressions which are homogeneous of order one in (b):

$$\beta_r = \zeta_{rk}(t_0)a_kb_n + b_r, \qquad \beta_n = \zeta_{nk}(t_0)a_kb_n + \beta_{no}b_n, \quad (r = 1,\dots,n-1)$$

where it is assumed that the functions y_i, z_i in (8:8) have been renumbered, if necessary, so that $\beta_{no} \neq 0$ and the functions $\varphi_i(a)$ in (8:10) are unchanged. The family (8:9) now contains the extremal arc E for $a_i = 0$, $b_r = \beta_{ro}$, $b_n = 1$, and the functions $y_i(t,a,b)$, $z_i(t,a,b)$ in (8:9) satisfy the relations (4:19). The proof given for Theorem 8:2 with a slight modification in the last set of equations in (8:11) is valid for the parameters (β) replaced by the expressions given above.

THEOREM 9:4. *If E is a non-singular extremal arc for the problem of Mayer satisfying the hypotheses of Theorem* 9:2, *then the function W described in Theorem* 8:3 *without the additive constant can be put into the form* $b_nU(t,y_1,\dots,y_n,c_1,\dots,c_{n-1})$, *and the number s described in the first part of Theorem* 8:5 *is at* *least equal to one. Conversely, if a function* $U(t,y_1,\dots,y_n,c_1,\dots,c_{n-1})$ *has continuous second partial derivatives with the determinant*

$$\begin{vmatrix} U_{y_i} \\ U_{y_i c_j} \end{vmatrix} \qquad (i = 1,\dots,n;\ j = 1,\dots,n-1)$$

different from zero for all points (t,y,c) *satisfying conditions of the form* (t,y) *in a region* $\mathcal{F}$, (c) *in a region* C, *and the sets* $(t,y,q) = (t,y,U_y)$ *corresponding to these arguments admissible, and if* U *satisfies the Hamilton-Jacobi equation* (8:13) *identically in* t, a, c, *then* $W = b_n U(t,y_1,\dots,y_n,c_1,\dots,c_{n-1})$ *for* $b_n \neq 0$ *is a function possessing the properties of the function* W *described in Theorem* 8:4. *If* U *also has continuous third derivatives, then the number* s *described in the last part of Theorem* 8:5 *is at least equal to one.*

The first part of the theorem follows from the conditions (4:18) and (9:1). The final statement in the theorem is a consequence of the homogeneity of the function H given in (4:18). To prove $s \geqq 1$, we use the identities $z_i H_{z_i z_j} = 0$, $H_{z_i z_j}(t,y,kz) = (1/k)H_{z_i z_j}(t,y,z)$, obtained by differentiating the equation (4:18) first with respect to z_i and the resulting equation with respect to k and with respect to z_j. With the help of these identities and $z_i = U_{y_i}$, it is found that

$$H_{z_i z_k}(y,W_y)W_{y_k}b_n = \frac{1}{b_n} H_{z_i z_k}(y,U_y)U_{y_k} = 0.$$

The number s is accordingly at least one by Theorem 8:5.

THEOREM 9:5. *Each transversal surface* $W(t,y) = c$ *of a field* $\mathcal{F}$, *determined by functions* $q_i(t,y)$, *contains an* $(n-1)$-*parameter family of extremals of the field defined by functions* of the form

$$(9{:}2)\qquad y_i(t,a_1,\ \dots,\ a_{n-1},c),\qquad z_i(t,a_1,\ \dots,\ a_{n-1},c).$$

At each point on the transversal surface, the equations

$$(9{:}3)\qquad q_i(t,y)y_{ia_k} = 0\qquad (i = 1,\dots,n;\ k = 1,\dots,n-1)$$

are identically satisfied by the functions $y_i(t,a,c)$ in (9:2).

Let the extremals of the field be given by the equations

$$(9{:}4)\qquad y_i = Y_i(t,a_1,\dots,a_n),\qquad z_i = Z_i(t,a_1,\dots,a_n)$$

where (a) lie in a region $\mathfrak{a}$ and are the values of Y_i at some point $t = t_0$ on the interval $t_1 \leqq t \leqq t_2$. Then, by the first part of Theorem 7:1, the transversal surfaces are determined by the equation $W(t,y)$ = constant, where

$$(9{:}5)\qquad W(t,y) = \int_{(t_0,y_0)}^{(t,y)} q_i(t,y)dy_i - H[t,y,q(t,y)]dt.$$

In view of Theorem 9:1 one has the identity

$$W[t,y(t,a)] = W(t_0,a).$$

Since the values W_{y_i} are not all zero, we can infer from the equation $W(t_0,a) = c$, that the parameters $a_1,\ \dots,\ a_n$ can be replaced by new parameters one of which is the constant c. We may denote these new parameters by $a_1,\ \dots,\ a_{n-1}$, c and hence express the extremals of the field by functions of the form 9:2. The equation (9:3) is obtained by differentiating the identity

$$W[t,y(t,a_1,\dots,a_{n-1},c)] = c$$

with respect to a_k and replacing W_{y_i} by its value $q_i(y)$. This proves Theorem 9:5.

10. Mayer fields and transversal surfaces for problem B. A Mayer field as defined by Bliss, is a region $\mathfrak{F}'$ of y-space containing only interior points and with which there is associated a set of slope-functions and multipliers $p_i(y)$, $\lambda_\beta(y)$ having continuous first partial derivatives in $\mathfrak{F}'$, defining elements (y,p) which lie in the region $\mathcal{R}_1$ and satisfy the equations $\phi_\beta = 0$ for every (y) in $\mathfrak{F}'$, and making the integral $I^* = \int F_{y_i'}\, dy_i$ independent of the path in $\mathfrak{F}'$ when the arguments of the function $F_{y_i'}$ are y, $p(y)$, $\lambda_\beta(y)$.

It should be noted that if a set $p_i(y)$, $\lambda_\beta(y)$ has the properties described above, then every set $k(y)\,p_i(y)$, $\lambda_\beta(y)$ for $k(y)$ a positive function with continuous derivatives for (y) in $\mathfrak{F}'$ also has these properties on account of the homogeneity of the functions F and ϕ_β and the fact that the region $\mathcal{R}_1$ contains the point (y,ky') for k positive if it contains the point (y,y'). In particular, I^* is unchanged, and it will be shown later that the extremals of the field remain the same. Since the function ϕ appearing in the function ϕ_0 for problem C has the properties prescribed for $k(y)$, the slope functions of the field $\mathfrak{F}'$ can be replaced by $p(y)/\phi[y,p(y)]$ and in this form they will satisfy the equation $\phi_0 = 0$. Furthermore, the corresponding cylindrical region $\overline{\mathfrak{F}}'$ of (t,y)-space with slope functions and multipliers $p_i(y)/\phi[y,p(y)]$, $\lambda_0 = 0$, $\lambda_\beta(y)$ is a Mayer field for problem C. Whenever the problems B and C are being compared it will be assumed that the functions $p_i(y)$ for problem B have been replaced by $p_i(y)/\phi[y,p(y)]$ although, as noted above, the results for problem B will be independent of this transformation. Since the solutions of the equations $y_i' = p_i(y)/\phi[y,p(y)]$ are extremals of the field $\overline{\mathfrak{F}}'$ with multipliers $\lambda_0 = 0$, $\lambda_\beta[y(t)]$, their projections in y-space are extremals for problem B with multipliers $\lambda_\beta[y(t)]$. By the method of Section 4, the initial values for the functions y_i can be expressed in terms of $n-1$ independent parameters. By replacing the parameter t by $T\phi(y,p)$ it is evident that every field $\mathfrak{F}'$ for problem B is simply covered by an $(n-1)$-parameter family of extremals satisfying the equations $y_i' = p_i(y)$. On an extremal arc E of a field $I^*(E) = I(E)$ since along such an arc the integrand of I^* reduces to F and $I(E) = \int_{t_1}^{t_2} F\,dt$.

Let $\bar{\mathfrak{D}}$ be a region of points (y,y',λ_β) with (y,y') interior to the region $\mathcal{R}_1$ described in Section 2 and so related to a region $\mathcal{N}$ of points (y,q) that to each point (y,q) interior to $\mathcal{N}$ there corresponds a unique solution $(y,y',\lambda_0 = 0,\lambda_\beta)$ of the equations (3:5) for problem C with (y,y',λ_β) interior to $\bar{\mathfrak{D}}$ and such that the determinant of the matrix R in (3:6) is different from zero. Let the region N for problem C, corresponding to the region N for problem A described in Section 6, be the region of points (t,y,q) such that (y,q) is in $\mathcal{N}$; similarly, let the region $\bar{D}$ for problem C, corresponding to the region $\bar{D}$ for problem A, be the region of points $(t,y,y',\lambda_0,\lambda_\beta)$ such that (y,y',λ_β) is interior to $\bar{\mathfrak{D}}$. The equations (4:3) with $z_i = q_i$ have as before single-valued solutions with continuous second derivatives and $\lambda_0 = H(y,q)$ = constant over the interior of N; furthermore every solution $y_i(t)$, $z_i = q_i(t)$ of equations (4:8) in the region N defines an extremal for problem C whose projection in the y-space is an extremal for problem B with $[y(t), q(t)]$ interior to the region $\mathcal{N}$. A set (y,q) interior to $\mathcal{N}$ will be called an admissible set.

A second definition of a Mayer field is a region $\mathfrak{F}$ of y-space containing only interior points with which are associated functions $q_i(y)$ having continuous first partial derivatives in $\mathfrak{F}$, defining elements (y,q) which lie in the region $\mathcal{N}$ and satisfy the equation $H(y,q) = 0$ for every (y) in $\mathfrak{F}$, and making the integral $I^* = \int q_i(y)\, dy_i$ independent of the path in $\mathfrak{F}$.

If $\mathfrak{F}$ is a field for problem B, then the corresponding cylindrical region of (ty)-space is a field $\bar{\mathfrak{F}}$ for problem C with the same functions $q_i(y)$ and the same invariant integral since $H[y,q(y)] = 0$. The region N for problem C is the region of points (t,y,q) such that (y,q) is in $\mathcal{N}$ and t is arbitrary.

THEOREM 10:1. *If the integral* I^* *is independent of the path in* $\mathfrak{F}$, *the equations*

$$\partial q_i/\partial y_k = \partial q_k/\partial y_i \qquad (i,k = 1, \ldots, n)$$

hold identically in the variables (y).

This theorem follows at once from Theorem 6:1 for problem C.

THEOREM 10:2. *Every solution* $y_i(t)$ *in the field* $\mathfrak{F}$ *of the equations*

$$(10:1) \qquad dy_i/dt = H_{z_i}[y,q(y)]$$

is an extremal with $z_i(t) = q_i[y(t)]$. *Through each point of the field there passes one and only one extremal.*

This theorem follows from Theorem 6:2 for problem C. Every solution $y_i(t)$ in the field $\bar{\mathfrak{F}}$ of the equations (10:1) with $z_i(t) = q_i[y(t)]$ satisfies the equations

$$(10:2) \qquad dy_i/dt = H_{z_i}(y,z), \quad dz_i/dt = -H_{y_i}(y,z), \quad H(y,z) = 0,$$

which are the canonical equations of the extremals. The projections in the y-space of the arcs satisfying the equations (10:2) are called *the extremals of the field* $\mathfrak{F}$. By the method of Section 4 the n parameters in the solutions of equations (10:1) can be expressed in terms of (n- 1) independent parameters in such a way that through each point of $\mathcal{F}$ there will pass one and only one extremal of the field.

A field $\mathcal{F}$ determined by functions $q_i(y)$ is a Mayer field as defined by Bliss with $p_i(y) = H_{z_i}[y,q(y)]$, $\chi_\beta = L_\beta[y,q(y)]$. A Mayer field defined by functions $p_i(y)$, $\chi_\beta(y)$ is a field according to the second definition provided that the sets $[y, q(y)]$ defined by points (y) in $\mathcal{F}'$, where $q_i(y) = F_{y_i'}[y,p(y),\chi(y)]$, are interior to the region $\mathcal{R}$ and satisfy the equation $H(y,q) = 0$.

Let $\mathcal{F}$ be a field defined by functions $q_i(y)$. An n-space $W(y) =$ constant is called a *transversal surface of the field* if at each point (y) on it the equation

$$q_i(y)dy_i = 0$$

is satisfied by every set of differentials dy_i which satisfy the equation

$$W_{y_i}dy_i = 0.$$

It is evident that a necessary and sufficient condition that an n-space $W(y) =$ constant be a transversal surface of a field $\mathcal{F}$ is that at each point on it in $\mathcal{F}$

$$W_{y_i} = kq_i(y)$$

where k is a proportionality factor.

THEOREM 10:3. *In every field $\mathcal{F}$ the Hilbert Integral defines a function $W(y)$ having continuous second partial derivatives in $\mathcal{F}$, satisfying the Hamilton-Jacobi partial differential equation*

$$H(y,W_y) = 0, \tag{10:3}$$

and such that the surfaces of the family $W(y)$ = constant *are the transversal surfaces of the field. Conversely, let* $W(y)$ *be a function which has continuous second partial derivatives and defines only admissible elements* $(y,q) = (y,W_y)$ *in a region* $\mathfrak{F}$ *of* y-*space. Then if* W *satisfies the Hamilton-Jacobi partial differential equation* (10:3), *the region* $\mathfrak{F}$ *is a field with*

$$q_i(y) = W_{y_i},$$

and the surfaces $W(y)$ = constant *are the transversal surfaces of the field.*

This theorem follows at once from Theorem 7:1 for problem C with the aid of the remarks preceding Theorem 10:1.

The following properties are easily established for a field $\mathfrak{F}$. In a field, $I^*(C_{12}) = W(y)\big|_1^2$ for any two points 1 and 2. Consequently, for every arc C_{12} on a transversal surface, $I^*(C_{12})$ is identically zero. From this it follows that, if the extremals of the field do not lie on the transversal surfaces, then two transversal surfaces intercept arcs on the extremals of the field along which the integral $\int z_i H_{z_i} dt$ has a constant value.

11. *Construction of special fields and the relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem* C. In this section it is shown how to construct fields for problem C for which the functions q_i are independent of t, and the special form which the theory developed in Section 8 assumes for this problem is given.

LEMMA 11:1. *If* $y_i(t)$, $z_i(t)$ *is a solution of equations* (4:8) *for problem* C *and the equation* $H(y,z)$ = constant, *then every set of the form* $\eta_i = cy_i'$, $\zeta_i = cz_i'$, *where* c *is a constant, is a solution of the canonical accessory equations* (8:7) *and the equation*

(11:1) $$H_{y_i}\eta_i + H_{z_i}\zeta_i = 0.$$

This statement is readily verified by direct substitution.

THEOREM 11:1. *If, in Theorem 8:2 for problem C, the conjugate system of solutions of equations* (8:7) *is of the form* y_i', z_i', $\eta_{ir}(t)$, $\zeta_{ir}(t)$ $(r = 1, \ldots, n-1)$ *and satisfy the equation* (11:1), *then by proper choice of the parameters the functions* q_i *for each field are independent of* t *and the Hamilton function* H(y,q) *associated with each field is a constant* χ_0. *The invariant integral described in Theorem* 8:3 *has the form*

(11:2) $$W(t,y,\beta,\chi_0) = \int_{(y_0)}^{(y)} q_i(y,\beta,\chi_0)dy_i - \chi_0 t = W(y,\beta,\chi_0) - \chi_0 t.$$

The theorem will be established by a method suggested by the proof of a theorem due to Bliss (V, 241). Let E' be a member for values $t_{10} \leqq t \leqq t_{20}$, a_{i0}, b_{i0} of a 2n-parameter family of extremals

(11:3) $$y_i = y_i(t-t_0,a,b), \qquad z_i = z_i(t-t_0,a,b)$$

where the parameters a_i, b_i are the values of the functions y_i, z_i, at $t = t_0$. By Theorem 4:3 the Hamilton function H(y,z) has a constant value H(a,b) along each extremal of the family (11:3). The functions y_i, y_{it}, z_i, z_{it} are assumed to have continuous first and second partial derivatives in a neighborhood of the values (t,a,b) on E'. The parameter t_0 appears only in the difference $t - t_0$ since the right members of equations (4:8) for problem C do not contain t explicitly. It is proposed to construct a Mayer family of fields containing E' for which the functions q_i associated with each field are independent of t. To do this, a function $U(\alpha,\beta)$ is defined by the equation

(11:4) $U(\alpha,\beta) = \frac{1}{2}\zeta_{ir}(t_0)\eta_{ij}(t_0)\alpha_r\alpha_j + \alpha_r\beta_r \quad (r = 1,\ldots,n-1)$

and the parameters a_i, b_i are determined as functions of α, β, χ_0 by means of the equations

(11:5) $$a_i(\alpha) = a_{i0} + \eta_{ir}(t_0)\alpha_r \qquad \begin{matrix}(i = 1,\ldots,n)\\(j,r = 1,\ldots,n-1)\end{matrix}$$

(11:6) $b_i\eta_{ij}(t_0) = U_{\alpha_j} = \zeta_{ir}(t_0)\eta_{ij}(t_0)\alpha_r + \beta_j$, $H[a(\alpha),b] = \chi_0$.

These equations are satisfied at the special values $(\alpha_j, \beta_j, \chi_0, a_i, b_i) = [0, b_{i0}\eta_{ij}(t_0), \chi_0(E'), a_{i0}, b_{i0}]$. For these special values the functional determinant of the left members of equations (11:6) with respect to the variables b_i is the determinant $|\eta_{ij}(t_0) \quad y_i'|$ which is different from zero along E' by hypothesis. Hence, equations (11:6) have solutions $b_i(\alpha,\beta,\chi_0)$ in a neighborhood of values $[\alpha_{r0}, \beta_{r0}, \chi_0(E')] = [0, b_{i0}\eta_{ir}(t_0), \chi_0(E')]$ which are such that $b_i[\alpha_0, \beta_0, \chi_0(E')] = b_{i0}$.

If the functions $a(\alpha)$, $b(\alpha,\beta,\chi_0)$ are substituted in equations (11:6) and the resulting identities are differentiated for α_r, β_r, it is seen that

(11:7) $b_{i\alpha_r}\eta_{ij}(t_0) = \zeta_{ir}(t_0)\eta_{ij}(t_0)$, $b_{i\alpha_r}H_{z_i} = -H_{y_i}\eta_{ir}(t_0)$

(11:8) $b_{i\beta_r}\eta_{ij}(t_0) = \delta_{rj}$, $b_{i\beta_r}H_{z_i} = 0$, $b_{i\chi_0}\eta_{ij}(t_0) = 0$,

$$b_{i\chi_0}H_{z_i} = 1,$$

where $\delta_{rr} = 1$ and $\delta_{rj} = 0$ if $r \neq j$. Since along E' the solutions y_i', z_i', $\eta_{ij}(t)$, $\zeta_{ij}(t)$ of equations (8:7) and (11:1) are assumed to form a conjugate system, it follows with the aid of equations (11:7) that along E'

$$[b_{i\alpha_r} - \zeta_{ir}(t_0)]\eta_{ij}(t_0) = 0, \quad [b_{i\alpha_r} - \zeta_{ir}(t_0)]y_i' = 0.$$

Since the determinant $|\eta_{ij}(t_0) \quad y_i'|$ is different from zero on E', the equations $b_{i\alpha_r} = \zeta_{ir}(t_0)$ must hold identically on E'. Furthermore, from equations (11:8) with $H_{z_i} = y_i'$, it is seen that at $t = t_0$ on E'

$$\begin{vmatrix} b_{i\beta_r} \\ b_{i\chi_0} \end{vmatrix} \cdot |\eta_{ij}(t_0) \quad y_i'| = \begin{vmatrix} \delta_{rj} & 0 \\ 0 & 1 \end{vmatrix} = 1, \tag{11:9}$$

and hence that the first determinant is different from zero.

The 2n-parameter family of extremals

$$y_i = Y_i(t-t_0, \alpha_r, \beta_r, \chi_0), \qquad z_i = Z_i(t-t_0, \alpha_r, \beta_r, \chi_0) \qquad (r = 1, \ldots, n-1) \tag{11:10}$$

obtained from equations (11:3) by substituting $a_i(\alpha), b_i(\alpha, \beta, \chi_0)$ for a_i and b_i contains the extremal E' for $(t_0, \alpha_r, \beta_r, \chi_0) = [t_{00}, 0, b_{i0}\eta_{ir}(t_0), \chi_0(E')]$ and can be shown to have the properties stated in the theorem.

By the method used in the proof of Theorem 8:2, it is readily established that at $t = t_0$ the variations of the family (11:10) with respect to the parameters are

$$\begin{aligned} &Y_{it_0} = -y_i', \quad Y_{i\alpha_r} = \eta_{ir}(t_0), \quad Z_{it_0} = -z_i', \quad Z_{i\alpha_r} = \zeta_{ir}(t_0), \\ &Y_{i\chi_0} = 0, \quad Y_{i\beta_r} = 0, \quad Z_{i\chi_0} = b_{i\chi_0}, \quad Z_{i\beta_r} = b_{i\beta_r}, \end{aligned} \tag{11:11}$$

and hence, that on E'

$$Y_{it_0} = -y_i', \quad Y_{i\alpha_r} = \eta_{ir}(t). \tag{11:12}$$

From the relations (11:11) and (11:9) it follows that the second determinant in (8:2) is different from zero at t_0 and consequently different from zero along E'. From the equations (11:12) it is seen that along E', $|Y_{it_0} \quad Y_{i\alpha_r}| = |-y_i' \quad \eta_{ir}(t)|$, and the latter determinant is different from zero. It follows that the first n equations in (11:10) can be solved for t_0, α_r as functions of t, β, χ_0. If these solutions $t_0(t, \beta, \chi_0)$, $\alpha_r(t, \beta, \chi_0)$ are substituted in the first n equations in (11:10), and the resulting identities are differentiated for t, one obtains the relation

$$0 = Y_i'(1 - t_0') + Y_{i\alpha_r}\alpha_r'.$$

Since the determinant $| Y_i' \; Y_{i\alpha_r} |$ is different from zero for $(t, t_0, \alpha, \beta, \chi_0)$ in a neighborhood $\mathcal{B}$ of the values on E', it follows that $t_0' = 1$, $\alpha_k' = 0$. Hence, the functions $t_0(t, \beta, \chi_0)$, $\alpha_r(t, \beta, \chi_0)$ are of the form

$$(11:13) \qquad t_0 = t - t(y, \beta, \chi_0), \qquad \alpha_r = \alpha_r(y, \beta, \chi_0)$$

for (β, χ_0) in $\mathcal{B}$ and (t, y) in a neighborhood $\mathcal{F}$ of the values on E'. With the aid of equations (11:13), functions $q_i(t, y, \beta, \chi_0)$ are defined by the equations

$$(11:14) \qquad q_i = Z_i[t(y, \beta, \chi_0), \alpha(y, \beta, \chi_0), \beta, \chi_0]$$

obtained by substituting the relations (11:13) in the last n equations in (11:10). It is clear that the functions q_i are independent of t. With the aid of the last equation in (11:8) it follows that

$$(11:15) \qquad H[y, q(y, \beta, \chi_0)] = \chi_0$$

is identically satisfied for each fixed set (β, χ_0) in $\mathcal{B}$ and (y) in $\mathcal{F}$. On the hyperplane $y_i = a_{io} + \eta_{ir}(t_0)\alpha_r$, on account of the relations (11:5), (11:6), (11:11) and (11:15) the integral I^* is independent of the path since

$$I^* = \int_C b_i \eta_{ir}(t_0)\, d\alpha_r - H[a(\alpha), b(\alpha, \beta, \chi_0)]\, dt = \int_C dU - \chi_0 t.$$

The integral I^* is therefore independent of the path in the whole neighborhood $\mathfrak{F}'$(IV, 106) and for each fixed set (β, γ_0) in $\mathfrak{B}$ the region $\mathfrak{F}'$ is a field with the functions $q_i(y, \beta, \gamma_0)$ defined in (11:14). The value of the integral I^* taken from a fixed point (t_0, y_0) in $\mathfrak{F}'$ to a variable point (t, y) has the form given in (11:2) on account of the definitions (11:14) and the identity (11:15).

THEOREM 11:2. *In each field* $\mathfrak{F}'$ *described in Theorem* 11:1, *the function* $\overline{W}$ *in* (11:2) *with an additive constant is a complete integral of the Hamilton-Jacobi differential equation*

$$\overline{W}_t + H(y, \overline{W}_y) = 0$$

and has the determinant $|\,\overline{W}_{y_i\beta_r} \quad \overline{W}_{y_i\gamma_0}\,|$ $(r = 1, \ldots, n - 1)$ *different from zero.*

This theorem follows from Theorem 11:1 and the first part of Theorem 7:1.

THEOREM 11:3. *If* $\overline{W}(t, y_1, \ldots, y_n, \beta_1, \ldots, \beta_{n-1}, \gamma_0)$ *has the properties described in Theorem* 8:4 *for problem* C *and is of the form* $\overline{W}(t, y, \beta, \gamma_0) = W(y, \beta, \gamma_0) - \gamma_0 t$, *then the equations*

$$W_{\beta_r}(y, \beta, \gamma_0) = W_{\beta_r}[a(\alpha), \beta, \gamma_0],$$

$$W_{\gamma_0}(y, \beta, \gamma_0) = W_{\gamma_0}[a(\alpha), \beta, \gamma_0] + t - t_0 \qquad (r = 1, \ldots, n-1)$$

define a 2n-*parameter family of extremals of the form* (11:10) *with properties similar to those described in Theorem* 4:1 *and which satisfy the equation*

$$H(y, W_y) = \gamma_0 \tag{11:16}$$

identically in t, y, β, γ_0, *the functions* $z_i(t - t_0, \alpha, \beta, \gamma_0)$ *associated* *with these extremals being defined by the equations*

$$z_i = W_{y_i}(y, \beta, \chi_0).$$

The functions $a_i(\alpha_1, \ldots, \alpha_{n-1})$ *are required to have continuous second partial derivatives for* (α) *in a region* $\mathcal{A}$ *and the determinant*

$$(11{:}17) \qquad \left| H_{z_i} \quad a_{i\alpha_r} \right| \qquad (i = 1, \ldots, n;\ r = 1, \ldots, n-1)$$

different from zero, where the arguments in H_{z_i} *are* $a_i(\alpha)$, $W_{y_i}[a(\alpha), \beta, \chi_0]$.

This theorem follows from Theorem 8:4 with $a_i = a_i(\alpha)$ and t_0 variable. With the aid of equation (11:16) and the fact that the determinant (11:17) is different from zero, it can be proved, using a method analogous to that used in the proof of Theorem 8:4, that the functional determinant of the variables (y, z) with respect to the parameters $(t_0, \alpha, \beta, \chi_0)$ is different from zero.

THEOREM 11:4. *If the last* s *of the parameters* (β) *do not appear in the functions* $Y_i(t - t_0, \alpha, \beta, \chi_0)$ *in* (11:10) *of Theorem* 11:1, *where* s *is one of the numbers* 1, ..., n-1, *then the function* $\overline{W}(t, y, \beta, \chi_0) = W(y, \beta, \chi_0) - \chi_0 t$ *described in Theorem* 11:2 *satisfies the additional equations*

$$(11{:}18) \qquad H_{z_i z_k} \overline{W}_{y_k \beta_\rho} = H_{z_i z_k} W_{y_k \beta_\rho} = 0 \qquad (\rho = n-s, \ldots, n-1).$$

Similarly, if the parameter χ_0 *does not appear in the functions* $Y_i(t - t_0, \alpha, \beta, \chi_0)$, *then the function* $\overline{W}$ *satisfies the equations*

$$(11{:}19) \qquad H_{z_i z_k} \overline{W}_{y_k \chi_0} = H_{z_i z_k} W_{y_k \chi_0} = 0.$$

If the function $\overline{W}(t, y, \beta, \chi_0) = W(y, \beta, \chi_0) - \chi_0 t$ *described in*

Theorem 11:3 has continuous third derivatives and satisfies the equations (11:18), *then* s *of the parameters* (β) *do not appear in the functions* $Y_i(t-t_0,\alpha,\beta,\gamma_0)$ *in* (11:10) *defining the extremals. Similarly, if the function* $\overline{W}$ *satisfies equation* (11:19) *then the parameter* γ_0 *does not appear in the functions* Y_i *in* (11:10).

This theorem follows from Theorem 8:5 for problem C with $a_i = a_i(\alpha)$ and t replaced by $t-t_0$.

THEOREM 11:5. *Let* $W(y_1,\dots,y_n,\beta_1,\dots,\beta_{n-1})$ *be a solution of the Hamilton-Jacobi equation* $W_t + H(y,W_y) = 0$ *for problem* C, *having continuous first and second derivatives for values* (y) *in* $\mathcal{F}$ *and* (β) *in* $\mathcal{B}$ *and such that the matrix* $\| W_{y_i\beta_r} \|$ $(r=1,\dots,n-1)$ *has rank* n - 1. *There exists a solution of the Hamilton-Jacobi equation for problem* C *of the form*

$$\overline{W}(t,y,\beta,\gamma_0) = W(y,\beta,\gamma_0) - \gamma_0 t$$

having continuous first and second derivatives for all values of (t,y,β,γ_0) *with* (y) *in* $\mathcal{F}$, (β) *in* $\mathcal{B}$ *and* γ_0 *near* $\gamma_0 = 0$, *at least if the domains* $\mathcal{F}$ *and* $\mathcal{B}$ *are taken sufficiently small. Moreover, the matrix* $\| W_{y_i\beta_r} \quad W_{y_i\gamma_0} \|$ *has rank* n *in this domain and* $\overline{W}(t,y,\beta,0) = W(y,\beta)$.

To prove this let $a_i(\alpha_1,\dots,\alpha_{n-1})$ be functions having continuous first and second derivatives and such that the determinant

$$\left| H_{z_i}(a(\alpha),W_y[a(\alpha),\beta]) \qquad a_{i\alpha_r} \right|$$

is different from zero and the values a_i are in $\mathcal{F}$. Such functions always exist if the domains $\mathcal{F}$ and $\mathcal{B}$ are sufficiently small. Let $b_i = b_i(\alpha,\beta,\gamma_0)$ be solutions of the equations

$$H[a(\alpha),b] = \gamma_0, \qquad b_i a_{i\alpha_r} = W_{y_i}[a(\alpha),\beta]a_{i\alpha_r}.$$

The function $\overline{W}(t,y,\beta,\gamma_0)$ may be determined as follows. Let

$$y_i(t-t_0,a,b), \qquad z_i(t-t_0,a,b)$$

be a 2n-parameter family of extremals for problem C containing the extremals of the field defined by the functions $q_i(y,\beta) = W_{y_i}(y,\beta)$ for values $b_i = W_{y_i}[a(\alpha),\beta]$, the values a_i, b_i being the values of the functions $y_i(t-t_0,a,b)$, $z_i(t-t_0,a,b)$ at $t = t_0$. As in the proof of Theorem 11:1, it is seen that the equations

$$y_i = y_i[t-t_0,a(\alpha),b(\alpha,\beta,\gamma_0)]$$

have solutions of the form $t_0 = t - t(y,\beta,\gamma_0)$, $\alpha_r = \alpha_r(y,\beta,\gamma_0)$ and that the function

$$\overline{W}(t,y,\beta,\gamma_0) = \int_{(t_0,y_0)}^{(t,y)} q_i dy_i - H(y,q)dt,$$

where $q_i(y,\beta,\gamma_0)$ are the values of the functions $z_i[t-t_0,a(\alpha),\beta,\gamma_0]$ when $t_0 = t - t(y,\beta,\gamma_0)$, $\alpha_r = \alpha_r(y,\beta,\gamma_0)$, has the properties described in the theorem.

12. Construction of fields and the relations between extremals and a complete integral of the Hamilton-Jacobi equation for problem B. An analogue of Theorem 8:2 for problem A is given in the following theorem.

THEOREM 12:1. *Let E be a non-singular extremal for problem B for which there exists a conjugate system Y_i', Z_i', η_{ir}, ζ_{ir} $(r = 1, \ldots, n-1)$ of solutions of the canonical accessory equations*

$$\eta_i' = H_{z_iy_k}\eta_k + H_{z_iz_k}\zeta_k, \qquad \zeta_i' = -H_{y_iy_k}\eta_k - H_{y_iz_k}\zeta_k$$

and the equation

$$H_{y_i}\eta_i + H_{z_i}\zeta_i = 0, \qquad (i,k = 1, \ldots, n)$$

with the determinant $| Y_i' \quad \eta_{ir}(t) |$ *different from zero along* E. *Then* E *is a member of a* (2n - 2)-*parameter family of extremals*

$$\text{(12:1)} \qquad \begin{aligned} Y_i &= Y_i(t, \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}), \\ Z_i &= Z_i(t, \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}) \end{aligned}$$

which are such that the matrices

$$\left\| Y_i' \quad Y_{i\alpha_r} \right\|, \quad \left\| \begin{matrix} Y_i' & Y_{i\alpha_r} & Y_{i\beta_r} \\ Z_i' & Z_{i\alpha_r} & Z_{i\beta_r} \end{matrix} \right\| \qquad (r = 1, \ldots, n-1)$$

have ranks n *and* 2n - 1 *respectively for* (t, α, β) *in a neighborhood of the values on* E. *Furthermore, for fixed values* (β) *in a region* $\mathcal{B}$, *the family* (12:1) *forms a Mayer field over a neighborhood* $\mathcal{F}$ *of points* (Y) *on* E, *and the functions* $q_i(Y, \beta)$ *associated with each field have the matrix*

$$\left\| q_{i\beta_r} \right\|$$

of rank n - 1. *The functions* $q_i(Y, \beta)$ *are defined by the equations*

$$\text{(12:2)} \qquad q_i(Y, \beta) = Z_i[t(Y, \beta), \alpha(Y, \beta), \beta],$$

and the functions $t(Y, \beta)$, $\alpha(Y, \beta)$ *are found by solving the first* n *equations in* (12:1).

This theorem follows from Theorem 11:1 with $\chi_0 = 0$. The field $\mathcal{F}$ for fixed values (β) in $\mathcal{B}$ is the projection in the y-space of the field $\mathfrak{F}'$ described in Theorem 11:1, and the extremals (12:1) are the projections in the y-space of those extremals in (11:10) for which $\chi_0 = 0$ and the parameter $t - t_0$ has been replaced by a new parameter t.

THEOREM 12:2. *Let* E *be a non-singular extremal for problem* B *for which the hypotheses of Theorem* 12:1 *are satisfied. Then there is a neighborhood of* E *in* y-*space in which, with an arbitrary additive constant, the function*

$$W(Y,\beta) = \int_{(Y_0)}^{(Y)} q_i(Y,\beta)\,dy_i$$

for q_i *defined by equations* (12:2) *is a complete integral of the Hamilton-Jacobi partial differential equation* $H(Y,W_Y) = 0$ *and has the matrix* $\| W_{y_i \beta_r} \|$ $(r = 1,\dots,n-1)$ *of rank* $n-1$.

This theorem follows from Theorem 12:1 for the case $\chi_0 = 0$ and $t - t_0$ replaced by a new parameter t.

THEOREM 12:3. *If* $W(y_1,\dots,y_n,\beta_1,\dots,\beta_{n-1})$ *has continuous second partial derivatives with the matrix*

$$\left\| W_{y_i \beta_r} \right\| \qquad (i = 1,\dots,n;\ r = 1,\dots,n-1) \tag{12:3}$$

of rank $n-1$ *for all points* (Y,β) *satisfying conditions of the form* (Y) *in a region* $\mathfrak{F}$ *in* y-*space* (β) *in a region* $\mathfrak{B}$, *and the sets* $(Y,q) = (Y,W_Y)$ *corresponding to these arguments admissible, and if furthermore* W *satisfies the Hamilton-Jacobi equation*

$$H(Y,W_Y) = 0 \tag{12:4}$$

identically in (Y,β), *then the equations*

$$W_{\beta_r}(Y,\beta) = W_{\beta_r}[a(\alpha),\beta] \tag{12:5}$$

define a $(2n-2)$-*parameter family of extremals*

$$(12:6) \qquad Y_i = Y_i[t, a(\alpha), \beta], \qquad Z_i = Z_i[t, a(\alpha), \beta]$$

with properties similar to those described in Theorem 4:6, *the functions* Z_i *associated with these extremals being defined by the equations*

$$Z_i = W_{Y_i} .$$

The functions $a_i(\alpha_1, \dots, \alpha_{n-1})$ *are required to have continuous second partial derivatives for* (α) *in a region* $\mathfrak{A}$ *and the determinant*

$$\left| H_{z_i} \quad a_{i\alpha_r} \right| \qquad (r = 1, \dots, n-1)$$

different from zero, where the arguments in H_{z_i} *are* $a_i(\alpha)$, $W_{Y_i}[a(\alpha), \beta]$.

This theorem follows at once from Theorem 11:3 with the aid of Theorem 11:5. The extremals (12:6) are the projections in the y-space of these extremals for problem C for which $\gamma_0 = 0$ and $t - t_0$ is replaced by a new parameter t.

THEOREM 12:4. *If the last* s *of the parameters* (β) *do not appear in the functions* $Y_i(t, \alpha, \beta)$ *in* (12:1) *of Theorem* 12:1, *where* s *is one of the numbers* $1, \dots, n-1$, *then the function* $W(Y, \beta)$ *described in Theorem* 12:2 *satisfies the additional equations*

$$(12:7) \qquad H_{z_i z_k} W_{Y_k \beta_\rho} = 0 \qquad (\rho = n-s, \dots, n-1).$$

If the function $W(Y, \beta)$ *described in Theorem* 12:3 *has continuous third derivatives and also satisfies equations* (12:7), *then the last* s *of the parameters* (β) *do not appear in the functions* $Y_i[t, a(\alpha), \beta]$ *in* (12:6) *defining the extremals.*

The first statement follows at once from the first part of Theorem 11:4 by setting $\chi_0 = 0$ and replacing the parameter $t - t_0$ by a new parameter t in the right members of the equations(11:10). With the aid of Theorem 11:5, the final statement in the theorem follows in a similar way from the last part of Theorem 11:4.

13. *Fields for the problem of Mayer in parametric form.* The theorems in this section are analogous to those for the non-parametric problem in Section 9.

THEOREM 13:1. *The integral* I^* *has the value zero along every extremal of a field* $\mathcal{F}$. *Furthermore, every extremal of a field lies on one of its transversal surfaces* $W(y)$ = constant *and the values* W_{y_i} *are not all zero.*

The first statement follows from the homogeneity of the function H given in (4:20) and the fact that $H[y,q(y)]$ is identically zero by the definition of a field for problem B. The final statement in the theorem is a consequence of the first part of Theorem 10:3 and the fact that W_{y_i} are the values of the functions z_i belonging to the extremals of the field.

THEOREM 13:2. *Let* E' *be an extremal arc for the problem of Mayer in non-parametric form corresponding to an extremal arc* E *for the problem of Mayer in parametric form and having the properties described in Theorem* 11:1. *Then by proper choice of of the parameters the family of extremals in* (11:10) *will satisfy the homogeneity relations* (4:19) *and the functions* $q_i(y,\beta,\chi_0)$ *described in* (11:14) *will satisfy the conditions*

$$q_i(y,k\beta,k\chi_0) = kq_i(y,\beta,\chi_0) \qquad (k \neq 0). \tag{13:1}$$

The proof is like that of Theorem 11:1 if the function $U(\alpha,\beta)$ in equation (11:4) is replaced by a suitable function of similar type that is homogeneous of order one in the parameters $\beta_1,\ldots,\beta_{n-1}$. It should be noted that not all of the expressions $b_{i0}\eta_{ir}(t_0)$ can be zero. This is a consequence of the condition (3:7) which reduces to the form $z_i y_i' = \chi_0 = 0$, the relation $z_i\eta_{ir}(t) =$ constant which follows from the equations $\Phi_\alpha = 0$ in (2:1) with the aid of the Euler-Lagrange equations (3:9)(III,307), and the hypothesis that the determinant $|\,y_i' \quad \eta_{ir}(t)|$ is different from zero along E'. It may be supposed, therefore, that $b_{i0}\eta_{i0}(t_0)$ is not zero. Then the function

$$U(\alpha,\beta) = \frac{1}{2}\zeta_{ir}(t_0)\eta_{ij}(t_0)\alpha_r\alpha_j\beta_1 + b_{i0}\eta_{i1}(t_0)\alpha_1\beta_1 + \alpha_s\beta_s$$

$$(r,j=1,\ldots,n-1;s=2,\ldots,n-1),$$

has the required properties. The condition (13:1) can be established by the method used to establish the condition (9:1) in the proof of Theorem 9:2.

THEOREM 13:3. Let E be a non-singular extremal arc for the problem of Mayer in parametric form having the properties described in Theorem 12:1. Then by proper choice of the parameters the family of extremals in (12:1) will satisfy the homogeneity relations (4:21) and the functions $q_i(Y,\beta)$ defined in (12:2) will satisfy the conditions

$$q_i(Y,k\beta) = kq_i(Y,\beta), \qquad (k \neq 0).$$

This theorem follows at once from Theorem 13:2 and Theorem 12:1.

THEOREM 13:4. If E is a non-singular extremal arc for the problem of Mayer satisfying the hypotheses of Theorem 13:3, *then the function W described in Theorem 12:2, without the additive constant, can be put into the form* $\beta_1 U(Y_1,\dots,Y_n,c_1,\dots,c_{n-s})$ *by proper choice of the parameters and the number* s *described in the first part of Theorem 12:4 is at least equal to one. Conversely, if a function* $U(Y_1,\dots,Y_n,c_1,\dots,c_{n-s})$ *has continuous second partial derivatives with the matrix*

$$\left\| \begin{matrix} U_{Y_i} \\ U_{Y_i c_j} \end{matrix} \right\| \qquad (i=1,\dots,n;\ j=1,\dots,n-2)$$

of rank $n-1$ *for all points* (Y,c) *satisfying conditions of the form* (Y) *in a region* $\mathfrak{F}$, (c) *in a region* $\mathcal{C}$, *and the sets* $(Y,q) = (Y,U_Y)$ *corresponding to these arguments admissible, and if* U *satisfies the* Hamilton-Jacobi *equation* (12:4) *identically in* (Y,c), *then* $W = \beta_1 U(Y_1,\dots,Y_n,c_1,\dots,c_{n-s})$ *for* $\beta_1 \neq 0$ *is a function possessing the properties of the function* W *described in Theorem* 12:3. *If* U *also has continuous third derivatives, then the number* s *described in the last part of Theorem* 12:4 *is at least equal to* 1.

The first part of the theorem is an immediate consequence of Theorem 13:3. The final statement of the theorem follows from the homogeneity of the function H given in (4:20) and its easily deduced consequence $z_i H_{z_i z_k} = 0$.

THEOREM 13:5. *Each transversal surface* $W(y) = c$ *of a field* $\mathfrak{F}$ *determined by functions* $q_i(y)$ *contains an* $(n-2)$-*parameter family of extremals of the field defined by functions of the form*

$$(13:3) \qquad y_i(t,\alpha_1,\dots,\alpha_{n-s},c), \qquad z_i(t,\alpha_1,\dots,\alpha_{n-s},c).$$

At each point on the transversal surface, the equations

$$(13:4) \qquad q_i(y)\,y_{i\alpha_k} = 0 \qquad (i=1,\dots,n;\ k=1,\dots,n-2)$$

are identically satisfied by the functions $y_i(t, \alpha, c)$ in (13:3).

By Theorem 10:3 the transversal surfaces of the field $\mathfrak{F}$ are defined by the equations $W(y)$ = constant, where

$$W(y) = \int_{y_0}^{y} q_i(y)dy_i,$$

in which y_0 is a fixed point of $\mathfrak{F}$ and $q_i(y)$ are the functions defining the field. The extremals of the field are given by functions

$$y_i(t, \alpha_1, \ldots, \alpha_{n-1}), \qquad z_i(t, \alpha_1, \ldots, \alpha_{n-1}),$$

having continuous first and second derivatives such that the determinant $| y_i' \ \ y_{i\alpha_r} |$ $(r = 1, \ldots, n-1)$ is different from zero in $\mathfrak{F}$. In view of Theorem 13:1, we have the identity

$$W[y(t, \alpha)] = W[y(t_0, \alpha)]$$

where t_0 is a fixed value of t. Moreover the values

$$W_{y_i}[y(t, \alpha)]y_{i\alpha_r} \tag{13:5}$$

are not all zero at $t = t_0$. To prove this, recall that the relations

$$z_i = W_{y_i}(y), \qquad z_i y_i' = z_i H_{z_i}(y, z) = H = 0,$$

hold with $y_i = y_i(t, \alpha)$, $z_i = z_i(t, \alpha)$. Since the z's are not all zero it follows that if the values (13:5) were all zero at $t = t_0$, the determinant $| y_i' \ \ y_{i\alpha_r} |$ would vanish at $t = t_0$ which is not the case.

By means of the equations $W[y(t_0, \alpha)] = c$ it is seen that the parameters $\alpha_1, \ldots, \alpha_{n-1}$ can be replaced by a new set of $n - 1$ parameters one of which is the constant c. We may accordingly suppose that $\alpha_{n-1} = c$. This proves the first part of the theorem. The second part is obtained by differentiating the equation $W[y(t, \alpha_1, \ldots, \alpha_{n-2}, c)] = c$ for α_k $(k = 1, \ldots, n-2)$ and replacing W_{y_i} by $q_i(y)$.

14. *Relation of Kneser field theory for the problem of Mayer in non-parametric form to the field theory developed in this paper.* Let E be a non-singular extremal which is a member for values $t_1 \leqq t \leqq t_2$, $a_{10}, \ldots, a_{n-1,0}$ of an $(n-1)$-parameter family of extremals

$$(14:1) \qquad y_i = y_i(t, a_1, \ldots, a_{n-1}), \qquad z_i = z_i(t, a_1, \ldots, a_{n-1})$$

which are such that along each extremal the matrix

$$(14:2) \qquad ||y_{ia_k}|| \qquad (i = 1, \ldots, n;\ k = 1, \ldots, n-1)$$

has rank $n-1$ and the equations

$$(14:3) \qquad z_i y_{ia_k} = F_{y_i'} y_{ia_k} = 0$$

are identically satisfied. *Then the family of extremals* (14:1) *defines a field in the sense of Kneser* (IX, 43).

Since along each extremal the function F for the problem of Mayer is identically zero, it follows readily with the aid of the Euler-Lagrange equations (3:5) that $F_{y_i'} y_{ia_k}$ = constant along each extremal (III, 307). Hence, if the constant is zero at the intersection of the extremal arcs in (14:1) with a sub-space defined by the equation $t = t(a_1, \ldots, a_{n-1})$, then the condition (14:3) is satisfied along each extremal of the family.

THEOREM 14:1. *Along each extremal of a Kneser field, the equation*

$$(F - y_i' F_{y_i'})dt + F_{y_i'} dy_i = 0$$

holds identically where $dy_i = y_i' dt + y_{ia_k} da_k$. *Furthermore, the extremals of a Kneser field lie on a surface which satisfies the Hamilton-Jacobi partial differential equation* $W_t + H(t, y, W_y) = 0$.

The first statement is an immediate consequence of the definition of a Kneser field and the fact that the function F is identically zero along each extremal. The analogue of this condition for a Mayer field in the sense of Bliss is the vanishing of the integral I^* along each extremal of a field. The second statement in the theorem is readily established by the method Kneser used for the parametric problem (IX, 44). The surface is obtained by eliminating the parameters from the first n equations in (14:1). This last result also follows readily from Theorems 9:1, 9:5 and the following theorem.

THEOREM 14:2. *A Mayer field in the sense of Bliss consists of a one-parameter family of Kneser fields.*

This theorem is an immediate consequence of Theorem 9:5. The condition (9:3) which must hold on each transversal surface of a Mayer field in the sense of Bliss is the condition (14:3) for a Kneser field.

THEOREM 14:3. *Every Kneser field can be imbedded in a one-parameter family of Kneser fields forming a Mayer field in the sense of Bliss, at least if the Kneser field is sufficiently small.*

Let E be a non-singular extremal arc imbedded for values $t_{10} \leqq t \leqq t_{20}$, α_{r0} in an $(n-1)$-parameter family of extremals

$$y_i = y_i(t, \alpha_1, \ldots, \alpha_{n-1}), \quad z_i = z_i(t, \alpha_1, \ldots, \alpha_{n-1})$$

which forms a Kneser field $\mathcal{F}'$ for (α) in a region $\mathcal{A}$. Set

(14:4) $$y_i(t_0,\alpha) = A_i(\alpha), \qquad z_i(t_0,\alpha) = B_i(\alpha)$$

and denote their values at $\alpha_r = \alpha_{ro}$ by a_{io}, b_{io}, respectively. From the definition of Kneser fields it follows that

(14:5) $$B_iA_{i\alpha_r} = 0 \qquad (r = 1,\ldots,n-1),$$

and the matrix $\|A_{i\alpha_r}\|$ has rank $n-1$. As was seen in Section 4 the extremal E can be imbedded for values $t_{10} \leqq t \leqq t_{20}$, a_{io}, b_{io} in a 2n-parameter family of extremals

(14:6) $$y_i = y_i(t,a,b), \qquad z_i = z_i(t,a,b)$$

of the type described in Theorem 4:1, where a_i, b_i are the values of y_i, z_i at $t = t_0$ and the parameters b_i do not vanish simultaneously.

Let 2n functions $a_i(\alpha,c)$, $b_i(\alpha,c)$ be defined by the equations

(14:7) $$a_i(\alpha,c) = A_i(\alpha) + cB_i(\alpha),$$

(14:8) $$b_i(A_{i\alpha_r} + cB_{i\alpha_r}) = 0, \qquad b_iB_i = B_i(\alpha\quad)B_i(\alpha\quad),$$

where $A_i(\alpha)$, $B_i(\alpha)$ are the functions in (14:4). With the aid of conditions (14:5), it is seen that the equations (14:7) and (14:8) are satisfied at the special values $(\alpha,c,a,b) = [\alpha_{ro}, 0, a_{io}, b_{io} = B_i(\alpha_{ro})]$. The functional determinant of the left members of equations (14:8) with respect to the parameters b_i is the determinant $|A_{i\alpha_r} + cB_{i\alpha_r} \quad B_i|$. For the special values given above, this determinant is different from zero. For, if it were equal to zero, there would exist constants d_r, d not all zero such that $d_rA_{i\alpha_r} + dB_i = 0$. Multiplying each of these equations by B_i and adding the equations, one finds that $d \neq 0$ on account of the conditions (14:5) and the fact that $B_iB_i \neq 0$. It follows that $d_rA_{i\alpha_r} = 0$ for constants d_r not all zero. This contradicts the hypothesis that the matrix $\| A_{i\alpha_r} \|$ has rank $n-1$. Hence, for (α, c) in a neighborhood of $(\alpha_{ro}, 0)$, the equations (14:8) have solutions $b_i = b_i(\alpha, c)$ which are such that $b_i(\alpha_{ro}, 0) = b_{io}$ and $b_i(\alpha, 0) = B_i(\alpha)$. The n-parameter family of extremals

$$\begin{aligned} y_i &= y_i[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Y_i(t, \alpha, c), \\ z_i &= z_i[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Z_i(t, \alpha, c), \end{aligned} \tag{14:9}$$

obtained from the family (14:6) by substituting $a_i(\alpha, c)$, $b_i(\alpha, c)$ for a_i, b_i, respectively, contains E for $\alpha_r = \alpha_{ro}$, $c = 0$, and will be shown to have the properties required for the theorem.

It will first be shown that the determinant $|Y_{i\alpha_r} \quad Y_{ic}|$ is different from zero along E. If this determinant were equal to zero at a point on E, there would exist constants d_r, d not all zero such that $d_rY_{i\alpha_r} + dY_{ic} = 0$. Multiplying each of these equations by Z_i and adding, one obtains the equation

$$d_rZ_iY_{i\alpha_r} + dZ_iY_{ic} = 0.$$

The expressions $Z_iY_{i\alpha_r}$, Z_iY_{ic} are constant along each extremal (III, 307) and since the first set vanishes at $t = t_o$ on account of condition (14:5), while the second set does not on account of the last group of equations in (14:8), it follows that $d = 0$. Hence, for constants d_r not all zero, $d_rY_{i\alpha_r} = 0$, which contradicts the hypothesis that the matrix (14:2) has rank $n-1$.

Furthermore, for values of α_r and each fixed value of c in a neighborhood of the values $(\alpha_{ro}, 0)$, the family (14:9) defines a Kneser field. For, the matrix $\| Y_{i\alpha_r} \|$ has rank $n-1$ and the expressions $Z_iY_{i\alpha_r}$ are identically zero on account of the first group of equations in (14:8).

Finally, the family (14:9) defines a Mayer field in a neighborhood $\mathcal{F}$ of the values (t,y) on E, with functions $q_i(t,y)$ defined by the equations

$$q_i(t,y) = Z_i[t, \alpha(t,y), c(t,y)],$$

where $\alpha_r(t,y)$, $c(t,y)$ are found by solving the first n equations in (14:9). To establish this statement it remains to be shown that the integral I^* is independent of the path in $\mathcal{F}$. On the hyperplane $t = t_0$, it is seen with the aid of the equations (14:7) and (14:8) that

$$\int_C q_i(t,y)dy_i = \int_C b_i(\alpha ,c)[A_{i\alpha_r} + cB_{i\alpha_r}]d\alpha_r + b_i(\alpha ,c)B_i dc$$

$$= B_i(\alpha\ )B_i(\alpha\ )c + \text{constant}.$$

The integral I^* is, therefore, independent of the path in the whole neighborhood $\mathcal{F}$ (IV, 106).

15. Relation of Kneser field theory for the problem of Mayer in parametric form to the field theory developed in this paper. Let E be a non-singular extremal which is the member for values $t_1 \leqq t \leqq t_2$, $a_{10},\dots,a_{n-2,0}$ of an $(n-2)$-parameter family of extremals

$$(15:1) \qquad y_i = y_i(t,a_1,\dots,a_{n-2}), \qquad z_i = z_i(t,a_1,\dots,a_{n-2})$$

which is such that along each extremal the matrix

$$(15:2) \qquad \left\|\, y_{it} \quad y_{ia_j} \,\right\| \qquad (i=1,\dots,n; j=1,\dots,n-2)$$

has rank $n-1$ and the equations

$$(15:3) \qquad F_{y_i'}\, y_{ia_j} = 0$$

are identically satisfied. Then *the family of extremals* (15:1) *forms a field in the sense of Kneser* (IX, 43)

As in the case of the non-parametric problem, the function F is identically zero, and it follows readily with the aid of the Euler-Lagrange equations (3:9) that $F_{y_i'} y_{ia_j}$ = constant along each extremal (III, 307). Accordingly, the expressions on the left of equations (15:3) vanish identically if they vanish at one point.

THEOREM 15:1. *Along each extremal of a Kneser field, the equation*

$$F_{y_i'}\, dy_i = 0$$

holds identically where $dy_i = y_i'\, dt + y_{ia_k} da_k$. *Furthermore, the extremals of a Kneser field lie on a surface which satisfies the Hamilton-Jacobi partial differential equation* $H(y,W_y) = 0$.

The first statement is an immediate consequence of the definition of a Kneser field and the fact that the function F is identically zero along each extremal. The analogue of this condition for a Mayer field in the sense of Bliss is the vanishing of the integral I^* along each extremal of a field. Kneser showed that the surface obtained by eliminating (t,a) from the first n equations in (15:1) satisfies the Hamilton-Jacobi equation. This last result also follows readily from Theorems 13:1, 13:5, and the following:

THEOREM 15:2. *A Mayer field in the sense of Bliss consists of a 1-parameter family of Kneser fields.*

This theorem is an immediate consequence of Theorem 13:5. The condition (13:4) which must hold on each transversal surface of a Mayer field in the sense of Bliss is the condition (15:3) for a Kneser field.

THEOREM 15:3. *Every Kneser field can be imbedded in a one-parameter family of Kneser fields forming a Mayer field in the* sense of Bliss, at least if the Kneser field is sufficiently small.

Let E be a non-singular extremal arc imbedded for values $t_{1o} \leqq t \leqq t_{2o}$, α_{ro} in an $(n-2)$-parameter family of extremals

$$y_i = y_i(t, \alpha_1, \ldots, \alpha_{n-2}), \qquad z_i = z_i(t, \alpha_1, \ldots, \alpha_{n-2})$$

which form a Kneser field $\mathfrak{F}'$ for (α) in a region $\mathfrak{A}$. Set

$$A_i(\alpha) = y_i(t_o, \alpha), \quad B_i(\alpha) = z_i(t_o, \alpha) \tag{15:4}$$

and denote their values at $t = t_o$, $\alpha_r = \alpha_{ro}$ by a_{io}, b_{io}, respectively. From the definition of Kneser fields it follows that

$$B_i A_{i\alpha_r} = 0 \qquad (r = 1, \ldots, n-2), \tag{15:5}$$

and the matrix $\| y_i' \quad A_{i\alpha_r} \|$ has rank $n-1$. As was seen in Section 4 the extremal E is imbedded for values $t_{1o} \leqq t \leqq t_{2o}$, a_{io}, b_{io} in a 2n-parameter family of extremals

$$y_i = y_i(t,a,b), \qquad z_i = z_i(t,a,b) \tag{15:6}$$

of the type described in Theorem 4:1 for problem C, where a_i, b_i are the values of y_i, z_i at $t = t_o$, the parameters b_i do not vanish simultaneously, and $H(a_{io}, b_{io}) = 0$.

Let 2n functions $a_i(\alpha, c)$, $b_i(\alpha, c)$ be defined by the equations

$$a_i(\alpha, c) = A_i(\alpha) + cB_i(\alpha) \tag{15:7}$$

$$b_i(A_{i\alpha_r} + cB_{i\alpha_r}) = 0, \quad b_iB_i = B_i(\alpha)B_i(\alpha), \quad H[a(\alpha,c),b] = 0 \qquad (r = 1, \ldots, n-2) \tag{15:8}$$

where $A_i(\alpha)$, $B_i(\alpha)$ are the functions in (15:4). With the aid of conditions (15:5), it is seen that the equations (15:7) and (15:8) are satisfied at the special values

$$[\alpha_r, c, a_i, b_i] = [\alpha_{ro}, 0, a_{io}, b_{io} = B_i(\alpha_{ro})].$$

The functional determinant of the left members of equations (15:8) with respect to the variables b_i is the determinant

$$\left| A_{i\alpha_r} + cB_{i\alpha_r} \quad B_i \quad y_i' \right|$$

with $H_{z_i} = y_i'$. For the special values given above, it can be shown that this determinant is different from zero by a method analogous to that used in the proof of Theorem 14:3 with the aid of the homogeneity condition $z_iH_{z_i} = H$. Therefore, the equations (15:8) have solutions $b_i(\alpha, c)$ for (α, c) in a neighborhood of the values $(\alpha_{ro}, 0)$ which are such that $b_i(\alpha_{ro}, 0) = b_{io}$ and $b_i(\alpha, 0) = B_i(\alpha)$.

The (n- 1)-parameter family of extremals

$$(15:9) \qquad \begin{aligned} y_i &= y_i[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Y_i(t, \alpha, c), \\ z_i &= z_i[t, A(\alpha) + cB(\alpha), b(\alpha, c)] = Z_i(t, \alpha, c), \end{aligned}$$

obtained from the family (15:6) by substituting $a_i(\alpha, c)$, $b_i(\alpha, c)$ for a_i, b_i, respectively, contains E for $\alpha_r = \alpha_{ro}$, $c = 0$, and will be shown to have the properties required for the theorem. By a method analogous to that used in the proof of Theorem 14:3 with the aid of the homogeneity condition $z_iH_{z_i} = H$, the conditions (15:5), (15:7) and (15:8), it is possible to show that the determinant $| Y_{it} \quad Y_{i\alpha_r} \quad Y_{ic} |$ is different from zero along E. Hence, the first n equations in (15:9) have solutions

$$(15:10) \qquad t = t(y), \qquad \alpha_r = \alpha_r(y), \qquad c = c(y) \quad (r=1,\dots,n-2)$$

which have continuous partial derivatives of the first order for (y) in a neighborhood of the values (y) on E.

The family (15:9) defines a Mayer field in a neighborhood $\mathcal{F}$ of the values (y) on E, with functions $q_i(y)$ defined by the equations

$$q_i(y) = Z_i[t(y), \alpha_r(y), c(y)],$$

the right members of which are obtained by substituting the functions (15:10) in the right members of the second group of equations in (15:9). To establish this statement, it is seen with the aid of the equations (15:7) and (15:8) that on the surface $y_i = A_i(\alpha) + cB_i(\alpha)$ the integral I^* assumes the form

$$\int_C q_i(y)dy_i = \int_C b_i(\alpha, c)[A_{i\alpha_r} + cB_{i\alpha_r}]d\alpha_r + b_i(\alpha, c)B_i dc$$

$$= B_i(\alpha)B_i(\alpha)c + \text{constant}.$$

It follows that the integral I^* is independent of the path in the whole neighborhood $\mathcal{F}$ (V, 215).

Moreover, for fixed values of c and (α) in a neighborhood of the values $(\alpha_{r0}, 0)$ belonging to E, the family (15:9) defines a Kneser field. This statement can be proved with the aid of the equations (15:7) and (15:8) by a method analogous to that used in the proof of Theorem 14:3.

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INDEX THEOREMS FOR THE PROBLEM OF BOLZA
IN THE
CALCULUS OF VARIATIONS

BY

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INDEX THEOREMS FOR THE PROBLEM OF BOLZA
IN THE CALCULUS OF VARIATIONS

1. Introduction. In the present paper we shall study index theorems of the problem in the calculus of variations determined by a functional of the form

$$J(\eta) = 2q(\eta) + \int_{x_1}^{x_2} 2\omega(\eta, \eta')dx \tag{1:1}$$

in $(x, \eta_1, \cdots, \eta_n)$-space subject to a set of end conditions

$$\Psi_\mu(\eta) = a_{\mu i}\eta_i(x_1) + b_{\mu i}\eta_i(x_2) = 0 \qquad (\mu = 1, \cdots, p < 2n) \tag{1:2}$$

and to a set of differential equations

$$\Phi_\nu(\eta, \eta') = S_{\nu i}(x)\eta_i + T_{\nu i}(x)\eta'_i = 0 \qquad (\nu = 1, \cdots, m < n). \tag{1:3}$$

In the functional (1:1) the integrand 2ω has the form

$$2\omega = P_{ik}(x)\eta_i\eta_k + 2Q_{ik}(x)\eta_i\eta'_k + R_{ik}(x)\eta'_i\eta'_k \qquad (i,k = 1, \cdots, n) \tag{1:4}$$

and 2q is a symmetric quadratic form in the end values $\eta_i(x_1)$, $\eta_i(x_2)$. A problem of this type is suggested by the study of the second variation for the general problems of Bolza and Lagrange and is usually called the accessory problem. In the present paper we shall generally denote this problem by the symbol P.

A large portion of the theory developed in the following pages is an extension to the problem of Bolza of the theory developed by Birkhoff and Hestenes (I, 200-243)* for the case in which the differential equations (1:3) are absent. Some of the results here given, particularly those in Section 3, were obtained also by Hestenes (VI) in an unpublished paper. Frequent use has been made of the types of proofs used in these papers. However, many modifications and simplifications have been introduced and in many cases the approach is entirely different from that of Birkhoff and Hestenes. Most of the simplifications here made are a consequence of an effective use of the relationships existing between problems and sub-problems. A connection between the index theory here presented and the index forms used by Morse is found in Section 7 below. An excellent discussion of boundary value problems and their relationships to index theorems in the calculus of variations has been given recently by Reid (XI).

In Section 2 certain fundamental definitions and assumptions essential to the analysis of later sections are given. In Section 3 the index of the problem P is defined as the dimensionality of a maximal linear subspace of the space of arcs under consideration on which $J(\eta)$ is negative definite. Various properties of the index are obtained, including its relationships to the natural isoperimetric conditions and the minimal sets used by Birkhoff and Hestenes.

In Section 4 the general notion of a sub-problem is introduced. By the use of such problems much of the theory of later sections is developed. In particular the well-known oscillation and comparison theorems given in Section 8 are proved by means of sub-problems. The concept of index of a problem with respect to a sub-problem is defined in Section 5.

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*Roman numerals in parenthesis refer to the bibliography at the end of the paper. The Arabic numerals refer to pages.

In Section 6 the one variable end point problem is discussed and in view of certain abnormality conditions which may arise it has been convenient to introduce the notion of focal intervals. These intervals play much the same role that focal points play in the simple problem. In particular it is shown that the index is equal to the number of focal intervals interior to the interval x_1x_2.

A special type of sub-problem is discussed in Section 7. This problem is obtained, in brief, by requiring the arcs associated with it to vanish at a certain prescribed set of points on x_1x_2. By properly choosing this set of points and making use of the properties of the index of a problem with respect to a sub-problem it is shown that the definition of index given in Section 3 is equivalent to the definition given by Morse in terms of broken extremal arcs.

In Section 9 there is a very brief discussion of the boundary value problem including the well-known result that the index is equal to the number of negative characteristic values of the associated boundary value problem. In Section 10 a pseudo-periodic problem is considered and several interesting notions including the frequency number are discussed. Also the equivalence of the index of repletion as defined by Morse and Pitcher (X) and the index of a problem with respect to a sub-problem is shown.

2. Preliminary definitions and assumptions. We shall assume in the quadratic form (1:4) that the functions $P_{ik} = P_{ki}$ are continuous and the functions Q_{ik} and $R_{ik} = R_{ki}$ are of class C' on the interval x_1x_2. By a function being of class C^n we mean that it is continuous and possesses continuous derivatives of the first n orders. Also we assume that the matrix $\| a_{\mu i},\ b_{\mu i} \|$ of the coefficients in (1:2) has rank p, that is, that the conditions (1:2) are linearly independent.

An arc $\eta_i(x)$ will be said to be of class D' if it is continuous and consists of a finite number of sub-arcs each of which is of class C'. Arcs of class D' will be called admissible arcs. Let $\mathcal{A}$ denote the set of admissible arcs and let $\mathcal{B}$ denote the set of all arcs η in $\mathcal{A}$ satisfying the differential equations (1:3). The functions $S_{\nu i}$ in these equations are assumed to be continuous and the functions $T_{\nu i}$ of class C'. Let $\mathcal{E}$ denote the sub class of $\mathcal{B}$ consisting of all arcs η in $\mathcal{B}$ satisfying the end conditions (1:2). It is to be noted that the set $\mathcal{E}$ is a linear set in the sense that if η and ξ are any two arcs in $\mathcal{E}$ and a, b are arbitrary real numbers, then the arc $a\eta + b\xi$ is also in $\mathcal{E}$. The sets $\mathcal{A}$ and $\mathcal{B}$ also form linear sets in this sense.

The Euler equations and transversality conditions for the accessory problem determined by (1:1) and the set of arcs are

$$(2:1)\qquad L_i(\eta, \eta', \eta'', \lambda) = \Omega_{\eta_i} - \frac{d}{dx}\Omega_{\eta_i'} = 0,$$

$$(2:2)\qquad \begin{aligned} T_{i1}(\eta, \lambda, \chi) &= q_{\eta i1} + \chi_\mu a_{\mu i} - \Omega_{\eta'_i}(x_1) = 0, \\ T_{i2}(\eta, \lambda, \chi) &= q_{\eta i2} + \chi_\mu b_{\mu i} + \Omega_{\eta'_i}(x_2) = 0, \end{aligned}$$

where $\Omega(\eta, \eta') = \omega(\eta, \eta') + \lambda_\nu(x)\, \Phi_\nu(\eta, \eta')$ and the symbol

η_{i1}, for example, stands for $\eta_i(x_1)$. A solution $\eta_i(x)$, $\lambda_\nu(x)$ of equations (1:3) and (2:1) where the functions $\eta_i(x)$ are of class C'' and the multipliers $\lambda_\nu(x)$ are of class C' is called an *accessory extremal* or, simply, an *extremal*. If the determinant

$$R = \begin{vmatrix} R_{ik} & T_{\nu i} \\ T_{\tau k} & 0 \end{vmatrix} \qquad (i,k = 1,\cdots,n;\ \tau,\ \nu = 1,\cdots,m)$$

is different from zero then an extremal may be expressed by the canonical variables η_i, ζ_i where $\zeta_i = \Omega_{\eta'_i}$. The canonical equations equivalent to (2:1) are

$$\eta'_i = A_{ik}\eta_k + B_{ik}\zeta_k, \qquad \zeta'_i = C_{ik}\eta_k - A_{ki}\zeta_k,$$

where the coefficients A_{ik}, B_{ik}, C_{ik} are continuous functions of x, (VII, 13-14). We shall assume throughout the following pages that R is different from zero and hence may use the canonical variables for an extremal whenever it seems more convenient to do so. An extremal which satisfies the transversality conditions (2:2) with a set of constants χ_μ will be called a *transversal extremal*. It will often be convenient to consider only the functions $\eta_i(x)$ associated with an extremal η_i, ζ_i. Such a set of functions η_i will be said to form an *extremal arc*. It is important to observe that the terms "extremal arc" and "extremal" here defined are used to denote different concepts, the extremal arc being the arc $\eta_i(x)$ $(x_1 \leqq x \leqq x_2)$ associated with the extremal $\eta_i(x)$, $\zeta_i(x)$ $(x_1 \leqq x \leqq x_2)$. It is also to be noted that an extremal arc may be associated with more than one extremal η_i, ζ_i. The functions $\eta_i(x)$ associated with a transversal extremal will be called a *transversal arc*.

The *Clebsch condition* will be said to hold on x_1x_2 if the

$$R_{ik}(x)\pi_i\pi_k \geqq 0 \tag{2:3}$$

is true for every set of constants $(\pi) \neq (0)$ satisfying the equations

$$T_{\nu i}(x)\pi_i = 0,$$

where R_{ik} and $T_{\nu i}$ are defined as in equations (1:4) and (1:3) respectively. We shall say that the strengthened Clebsch condition holds if the condition (2:3) holds without the equality sign.

It may be noted that the accessory problem P can be formulated in different ways. For example the functional $J(\eta)$ may be of the form

$$J(\eta, u) = b_{h\ell}u_h u_\ell + \int_{x_1}^{x_2} 2\omega(\eta, \eta')dx \qquad (h, \ell = 1, \cdots, r)$$

and subject to the conditions (1:3) and end conditions

$$\eta_i(x_s) - \overset{s}{c}_{ih}u_h = 0 \qquad (i = 1, \cdots, n;\ h = 1, \cdots, r;\ s = 1, 2),$$

where the matrix $\|b_{h\ell}\|$ is symmetric and u_h, $\overset{s}{c}_{ih}$ are constants. That this problem is equivalent to a problem of the type defined in Section 1 in a space of $n + r + 1$ dimensions is seen by adding the differential equations $u'_h = \eta'_{n+h} = 0$.

3. Index of $J(\eta)$ on $\mathcal{E}$ and minimal sets. Let $\xi_i(x)$ be an arc in $\mathcal{E}$ and define $J(\xi, \eta)$ to be the bilinear functional

$$\begin{aligned} J(\xi, \eta) = {} & \xi_{i1} q \eta_{i1} + \xi_{i2} q \eta_{i2} \\ & + \int_{x_1}^{x_2} (\xi_i \omega_{\eta_i} + \xi'_i \omega_{\eta'_i})dx. \end{aligned} \tag{3:1}$$

The condition

$$J(\xi, \eta) = 0$$

will be called a *natural isoperimetric condition for the functional* J(η) *on* $\mathcal{E}$.

We now prove the following

THEOREM 3:1. *An arc* η *in* $\mathcal{B}$, *that is, an admissible arc satisfying the differential equations* (1:3), *is a transversal arc if and only if* η *satisfies the condition* J(ξ, η) = 0 *for every arc* ξ *in* $\mathcal{E}$.

For suppose η is a transversal arc and let λ_ν, χ_μ be a set of multipliers associated with η such that the equations (2:1) and (2:2) are satisfied. Since J(ξ, η) is expressible in the form

$$J(\xi,\eta) = \xi_{i1}q_{\eta i1} + \xi_{i2}q_{\eta i2} + \int_{x_1}^{x_2}\left[\xi_i \Omega_{\eta_i} + \xi'_i \Omega_{\eta'_i} + \lambda_\nu \Phi_\nu(\xi,\xi')\right]dx$$

we have, after adding $\chi_\mu \Psi_\mu(\xi)$ to both sides and integrating by parts, that

$$\begin{aligned} J(\xi,\eta) + \chi_\mu \Psi_\mu(\xi) = {} & \xi_{i1}\left[q_{\eta i1} + \chi_\mu a_{\mu i} - \zeta_i(x_1)\right] \\ & + \xi_{i2}\left[q_{\eta i2} + \chi_\mu b_{\mu i} + \zeta_i(x_2)\right] \\ & + \int_{x_1}^{x_2}\left[\xi_i L_i(\eta) + \lambda_\nu \Phi_\nu(\xi,\xi')\right]dx, \end{aligned}$$

where $\zeta_i = \Omega_{\eta'_i}$ and $L_i(\eta)$ are the functions defined in (2:1). If ξ is any arc in $\mathcal{E}$ then $\Psi_\mu(\xi) = 0$ and the constants χ_μ may be chosen so that the quantities in the first two brackets are equal respectively to the functions T_{i1}, T_{i2} in (2:2). With the help of equations (2:1) it follows that J(ξ, η) = 0.

Conversely, suppose J(ξ, η) = 0 for all arcs ξ in $\mathcal{E}$. Let ξ_α (α = 1,···,r) be a set of arcs in $\mathcal{B}$ such that the matrix $\|\Psi_\mu(\xi_\alpha)\|$ has maximum rank r. It follows that the matrix

$$\left\| \begin{matrix} J(\xi_\alpha, \eta), & J(\xi, \eta) \\ \Psi_\mu(\xi_\alpha), & \Psi_\mu(\xi) \end{matrix} \right\|$$

also has rank r for all arcs ξ in ϑ. To show this we note that there exist constants a_α such that $\Psi_\mu(\xi + a_\alpha \xi_\alpha) = 0$. Hence the arc $\xi + a_\alpha \xi_\alpha$ is in $\mathcal{E}$ and $J(\xi + a_\alpha \xi_\alpha, \eta) = 0$. The above matrix accordingly has rank r for every arc ξ in ϑ. It now follows that there exist constants b_μ such that the equation

$$J(\xi, \eta) + b_\mu \Psi_\mu(\xi) = 0$$

holds for all arcs ξ in ϑ. The remaining part of the argument is similar to that given by Bliss in the proof of the Multiplier Rule for the problem of Lagrange and will not be given here (III, 23-26).

We now establish the result

THEOREM 3:2. _The number of linearly independent arcs in a maximal set of arcs in $\mathcal{E}$, every proper linear combination η of which gives $J(\eta)$ a negative value, is the same for every such set._

In order to prove this let ξ_α $(\alpha = 1,\cdots,k)$, η_γ $(\gamma = 1,\cdots,k')$ be two such maximal sets of arcs in $\mathcal{E}$. If $k < k'$ there exist constants c_γ, not all zero, such that $J(\xi_\alpha, c_\gamma \eta_\gamma) = 0$. It then follows that every linear combination η of the arcs ξ_α, $c_\gamma \eta_\gamma$ gives $J(\eta)$ a negative value which contradicts the assumption on the ξ_α. By symmetry k cannot be greater than k'. Hence k = k'.

If the number of linearly independent arcs in a maximal set of arcs in $\mathcal{E}$ every proper linear combination η of which gives $J(\eta)$ a negative value is finite, then this number is defined to be the *index* of $J(\eta)$ on $\mathcal{E}$. If the number of such arcs is not finite the index is said to be infinite. In Section 7 it will be shown that if the strengthened Clebsch condition holds then the index is always finite. We shall assume that the index is finite in this and following sections except, of course, in the proof of the above fact.

A set of k natural isoperimetric conditions

$$J(\xi_\alpha, \eta) = 0 \qquad (\alpha = 1,\cdots,k) \tag{3:2}$$

will be called a *proper set* if the determinant $|J(\xi_\alpha, \xi_\beta)|$ is different from zero. The set (3:2) will be called a *minimal set* if the inequality $J(\eta) \geqq 0$ is true for every arc η in $\mathcal{E}$ satisfying these conditions and if there exists no set of the form (3:2) with fewer conditions having this property. This differs from the definition of a minimal set as stated by Birkhoff and Hestenes, however, an examination of the proofs of their theorems shows that the definition used is that stated here. If the set (3:2) is a minimal set then the set $J(\bar{\xi}_\alpha, \eta) = 0$ is also a minimal set where $\bar{\xi}_\alpha = \xi_\beta A_{\alpha\beta}$, the A's being constants with $|A_{\alpha\beta}| \neq 0$. As a consequence of this fact we may suppose that the arcs ξ_α are chosen so that

$$J(\xi_\alpha, \xi_\beta) = 0 \qquad (\alpha \neq \beta;\ \alpha, \beta = 1,\cdots,k). \tag{3:3}$$

Concerning minimal sets we have the following useful theorems.

THEOREM 3:3. *Let* $J(\xi_\alpha, \eta) = 0$ $(\alpha = 1,\cdots,k)$ *be a proper minimal set defined by arcs* ξ_α *in* $\mathcal{E}$. *If there exists an arc* ξ *in* $\mathcal{E}$ *such that* $J(\xi) = J(\xi_\alpha, \xi) = 0$, *then* ξ *is a transversal arc*.

The theorem follows from Theorem 3:1 when it is shown that $J(\xi, \eta) = 0$ for every arc η in $\mathcal{E}$. Let η be any arc in $\mathcal{E}$. If $J(\xi_\alpha, \eta) = 0$ then the equations

$$J(\xi_\alpha, \eta + b\xi) = J(\xi_\alpha, \eta) + bJ(\xi_\alpha, \xi) = 0$$

hold for all b. Consequently the inequality

$$0 \leqq J(\eta + b\xi) = J(\eta) + bJ(\eta, \xi)$$

is true for all b and hence $J(\eta, \xi) = 0$. If $J(\xi_\alpha, \eta) \neq 0$ then, since $|J(\xi_\alpha, \xi_\beta)| \neq 0$, we can select constants a_α $(\alpha = 1, \cdots, k)$ such that the arc $\bar{\xi} = \eta - \xi_\alpha a_\alpha$ satisfies $J(\xi_\alpha, \bar{\xi}) = 0$. By the preceding argument

$$0 = J(\bar{\xi}, \xi) = J(\eta, \xi) - a_\alpha J(\xi_\alpha, \xi) = J(\eta, \xi).$$

Hence the theorem follows.

THEOREM 3:4. *If the set* (3:2) *is a minimal set defined by arcs* ξ_α $(\alpha = 1, \cdots, k)$ *in* $\mathcal{E}$, *then no arc of the form* $\xi = \xi_\alpha a_\alpha$, *where the a's are constants not all zero, is a transversal arc. Moreover, the inequality* $J(\xi) \leqq 0$ *is true for every arc of this form, and if the set* (3:2) *is a proper minimal set the strict inequality* $J(\xi) < 0$ *holds.*

To prove the first statement of the theorem suppose there exists an arc in $\mathcal{E}$ of the form $\xi = \xi_\alpha a_\alpha$ which is a transversal arc. At least one of the a_α, say a_k, is different from zero. Now let η be any arc in $\mathcal{E}$ satisfying the conditions $J(\xi_\beta, \eta) = 0$ $(\beta = 1, \cdots, k-1)$. Since

$$0 = J(\xi, \eta) = J(\xi_\alpha a_\alpha, \eta) = J(\xi_k a_k, \eta) \qquad (\alpha = 1, \cdots, k)$$

it follows that $J(\eta) \geqq 0$. The set (3:2), therefore cannot be minimal for the set $J(\xi_\beta, \eta) = 0$ contains fewer then k conditions.

Now suppose that the arcs ξ_α have been chosen so that the conditions (3:3) hold and that one of the numbers $J(\xi_\alpha, \xi_\alpha)$, say $J(\xi_k, \xi_k)$, is positive. Let η be an arc in $\mathcal{E}$ satisfying only the first $(k-1)$ conditions (3:2) and having $J(\eta) < 0$. There exists a constant b such that

$$J(\xi_k, \eta) = bJ(\xi_k, \xi_k) = bJ(\xi_k).$$

The arc $\eta - b\xi_k$ then satisfies all the conditions (3:2). But we have

$$\begin{aligned} J(\eta - b\xi_k) &= J(\eta) - 2bJ(\xi_k, \eta) + b^2 J(\xi_k) \\ &= J(\eta) - b^2 J(\xi_k) < 0 \end{aligned}$$

which is contrary to our assumption that the set (3:2) is a minimal set. If $|J(\xi_\alpha, \xi_\beta)| \neq 0$ and the conditions (3:3) hold, then clearly every arc of the form $\xi = a_\alpha \xi_\alpha$ has $J(\xi) < 0$.

As a converse to the first part of this theorem we have

THEOREM 3:5. *Let* ξ_α $(\alpha = 1, \cdots, k)$ *be a maximal set of arcs in* $\mathcal{E}$ *having the following properties*: (1) *every arc of the form* $\xi = \xi_\alpha a_\alpha$, *where the* a's *are constants not all zero, satisfies the inequality* $J(\xi) \leqq 0$, (2) *no arc of the form* $\xi = \xi_\alpha a_\alpha$ $(a) \neq (0)$ *is a transversal arc*. *Then the set* $J(\xi_\alpha, \eta) = 0$ *forms a minimal set.*

For suppose there exists an arc η in $\mathcal{E}$ satisfying the conditions $J(\xi_\alpha, \eta) = 0$ and having $J(\eta) < 0$. Then every linear combination ξ of the $(k+1)$ linearly independent arcs ξ_α, η gives $J(\xi)$ a negative value and no proper linear combination of these arcs is a transversal arc since $J(\eta) < 0$. This contradicts the assumption on the ξ_α. Hence every arc η in $\mathcal{E}$ satisfying $J(\xi_\alpha, \eta) = 0$ has $J(\eta) \geqq 0$. Now suppose there exist linearly independent arcs η_γ $(\gamma = 1,\cdots,k')$ such that the conditions $J(\eta_\gamma, \eta) = 0$ $(\gamma = 1,\cdots, k' < k)$ form a proper minimal set. Since $k' < k$ there exist solutions $(b) \neq (0)$ of the equations $J(\eta_\gamma, \xi_\alpha)b_\alpha = 0$. Hence the arc $\xi = \xi_\alpha b_\alpha$ has $J(\xi) \geqq 0$, but by assumption $J(\xi) \leqq 0$. Hence $J(\xi) = 0$. As a consequence of Theorem 3:3 the arc ξ is a transversal arc contrary to our assumption. The statement of the theorem now follows.

As a converse to the last statement of Theorem 3:4 we have

THEOREM 3:6. *If the variations* ξ_α $(\alpha = 1,\cdots,k)$ *form a maximal set of linearly independent arcs in* $\mathcal{E}$, *every proper linear combination* η *of which gives* $J(\eta)$ *a negative value, then the set* $J(\xi_\alpha, \eta) = 0$ *is a proper minimal set.*

By the type of argument used in the proof of Theorem 3:5 it follows that every arc η in $\mathcal{E}$ satisfying the conditions $J(\xi_\alpha, \eta) = 0$ has $J(\eta) \geqq 0$. Moreover there can exist no minimal set with fewer than k conditions for then there would exist an arc of the form $\xi = \xi_\alpha a_\alpha$ having $J(\xi) \geqq 0$. The set $J(\xi_\alpha, \eta) = 0$ is proper for if it were not the determinant $|J(\xi_\alpha, \xi_\beta)|$ would be equal to zero and there would exist solutions $(b) \neq (0)$ of the equations $J(\xi_\alpha, \xi_\beta)b_\alpha = 0$ which is impossible.

The following theorem gives a useful relationship between minimal sets and the index of $J(\eta)$ on $\mathcal{E}$.

THEOREM 3:7. *If the set* $J(\xi_\alpha, \eta) = 0$ $(\alpha = 1,\cdots,k)$ *is a minimal set, then* k *is the index of* $J(\eta)$ *on* $\mathcal{E}$.

For, assume that the index of $J(\eta)$ on $\mathcal{E}$ is k' and let η_γ $(\gamma = 1,\cdots,k')$ be a maximal set of linearly independent arcs in $\mathcal{E}$ such that every arc of the form $\eta = \eta_\gamma a_\gamma$, where the a's are constants not all zero, satisfies the inequality $J(\eta) < 0$. Then every solution ξ of $J(\eta_\gamma, \xi) = 0$ has $J(\xi) \geqq 0$. As a consequence of the definition of minimal sets k cannot be greater than k'. If $k' > k$ then there exist solutions $(a) \neq (0)$ of the equations $J(\xi_\alpha, \eta_\gamma)a_\gamma = 0$ $(\alpha = 1,\cdots,k)$. The set $J(\xi_\alpha, \eta) = 0$ being a minimal set implies that $J(\eta_\gamma a_\gamma) \geqq 0$ contrary to the choice of the η_γ. Hence $k = k'$ and the conclusion of the theorem is immediate.

COROLLARY 1. *The number of conditions in a minimal set is always the same.*

COROLLARY 2. *If the conditions* $J(\xi_\alpha, \eta) = 0$ $(\alpha = 1,\cdots,k)$ *form a minimal set, then there exists a proper minimal set containing the same number of conditions.*

COROLLARY 3. *The index of* $J(\eta)$ *on* $\mathfrak{E}$ *is equal to the number of arcs* ξ_α *in a maximal set of arcs in* $\mathfrak{E}$ *every linear combination* ξ *of which has* $J(\xi) \leqq 0$ *but no proper linear combination of which is a transversal arc.*

This follows from Theorems 3:5 and 3:7.

4. *Sub-problems.* In this section we shall let P be a problem determined by a functional $J(\eta)$ and a set $\mathfrak{E}$ of admissible arcs defined as in Section 2. If P^* is a problem determined by the functional $J(\eta)$ and a set of admissible arcs $\mathfrak{E}^*$ where $\mathfrak{E}^*$ is a linear subset of $\mathfrak{E}$, then P^* is said to be a *sub-problem of the problem* P. Sub-problems P^* may be obtained, for example, by adding a set of end conditions to the conditions for the problem P. As an illustration consider the fixed end point problem determined by $J(\eta)$ and a set of arcs vanishing at x_1 and x_2. It may be obtained from the problem P by adding conditions of the form $\eta_i(x_1) = \eta_i(x_2) = 0$. We shall denote the fixed end point problem by P° and the set of arcs vanishing at x_1 and x_2 by $\mathfrak{E}^\circ$. Sub-problems P^* of P may also be obtained by adding a set of natural isoperimetric conditions to the conditions for P, or by

defining $\mathcal{E}^*$ to be the set of all arcs in $\mathcal{E}$ which satisfy the conditions $A_{is}\eta_i(x^s) = 0$ ($s = 1,\cdots, r < n$) where the points x^s are on the interval x_1x_2. In particular we have the sub-problem determined by the set of arcs in $\mathcal{E}$ which satisfy the conditions $\eta_i(x^s) = 0$. Unless otherwise stated we shall consider the sub-problem P^* to be such that the end conditions $\Psi_\mu = 0$ ($\mu = 1,\cdots,p$) for P are linearly dependent on the end conditions $\Psi^*_{\bar\mu} = 0$ ($\bar\mu = 1,\cdots,p^*$) for P^*, that is, a sub-problem obtained by adding a set of end conditions to those for P.

Before considering sub-problems further we establish the following result:

THEOREM 4:1. A necessary and sufficient condition that the end points of an arc η in $\mathcal{E}$ can be joined by an extremal arc $\bar\eta$ is that $J(\eta, \xi) = 0$ for all extremal arcs ξ in $\mathcal{E}^0$, that is, for all extremal arcs vanishing at x_1 and x_2.

The condition is necessary, for let $\bar\eta$ be an extremal arc joining the end points of η. Then we have

$$0 = J(\eta - \bar\eta, \xi) = J(\eta, \xi) - J(\bar\eta, \xi) = J(\eta, \xi)$$

for any extremal arc ξ in $\mathcal{E}^0$.

In order to show that the condition is sufficient let ξ_α ($\alpha = 1,\cdots,k$) and a set of functions $\zeta_\alpha = \Omega_{\xi'_\alpha}$, form a maximal set of linearly independent extremals in $\mathcal{E}^0$. Then we have

$$(4:1)\qquad J(\xi_\alpha, \eta) = \eta_i(x_2)\zeta_{i\alpha}(x_2) - \eta_i(x_1)\zeta_{i\alpha}(x_1) = 0.$$

It is well known that for any two extremals η, ζ and $\bar\eta, \bar\zeta$ the expression $\eta\bar\zeta - \bar\eta\zeta$ is a constant and hence has the same value at x_1 as at x_2 (II, 738). As a consequence of this fact and the relations $\xi_\alpha(x_1) = \xi_\alpha(x_2) = 0$ it follows that the condition (4:1) holds when the values $\eta_i(x_1)$, $\eta_i(x_2)$ are the end values of an extremal arc η. Let ξ_γ, ζ_γ ($\gamma = k+1,\cdots,2n$) be a set of $(2n-k)$ extremals such that the set ξ_β, ζ_β ($\beta = 1,\cdots,2n$) form a maximal set of linearly independent extremals. Since the extremals ξ_α, ζ_α ($\alpha = 1,\cdots,k$) form a maximal set of linearly independent extremals vanishing at x_1 and x_2, the matrix

$$\left\|\begin{matrix} \xi_{i\gamma}(x_2) \\ \xi_{i\gamma}(x_1) \end{matrix}\right\| \qquad (\gamma = k+1,\cdots,2n)$$

must have rank $2n-k$. The columns of this matrix accordingly form a set of $2n-k$ linearly independent solutions $\eta_i(x_2)$, $\eta_i(x_1)$ of the equations (4:1). Moreover the equations (4:1) are linearly independent since otherwise the extremals ξ_α, ζ_α could not be linearly independent. It follows that the end values $\eta_i(x_1)$, $\eta_i(x_2)$ of an arc η satisfying equations (4:1) are expressible in the form $\eta_i(x_1) = \xi_{i\gamma}(x_1)c_\gamma$, $\eta_i(x_2) = \xi_{i\gamma}(x_2)c_\gamma$ where the c's are constants and hence the theorem is established.

COROLLARY 1. *A necessary and sufficient condition that the end points of an arc* η *in* $\mathcal{E}$ *can be joined by an extremal arc* $\bar{\eta}$ *is that* $J(\eta, \xi_\alpha) = 0$ *where the* ξ_α *form a maximal set of linearly independent extremal arcs in* $\mathcal{E}^{\circ}$ *no proper linear combination of which is a transversal arc for* P.

This follows readily from Theorem 4:1 since the condition $J(\eta, \xi) = 0$ is surely satisfied for all arcs ξ in $\mathcal{E}^{\circ}$ which are transversal arcs for P.

COROLLARY 2. *A necessary and sufficient condition that an arc* η *in* $\mathcal{E}$ *be expressible in the form* $\eta = \eta_1 + \eta_2$, *where* η_1 *is an extremal arc in* $\mathcal{E}$ *and* η_2 *is an arc in* $\mathcal{E}^{\circ}$, *is that* $J(\eta, \xi) = 0$ *for all extremal arcs* ξ *in* $\mathcal{E}^{\circ}$. *Moreover* $J(\eta) = J(\eta_1) + J(\eta_2)$ *in this case.*

As a consequence of Theorem 4:1 we have

THEOREM 4:2. *Let* η *be any arc in* $\mathcal{E}$ *whose end points cannot be joined by an extremal arc. There exists an arc* ξ *in* $\mathcal{E}$ *joining the end points of* η *such that* $J(\xi) < 0$.

For, if the end points of η cannot be joined by an extremal arc there exists an extremal arc $\bar{\eta}$ in $\mathcal{E}^*$ such that $J(\eta, \bar{\eta}) \neq 0$. The arc $\xi = \eta + b\bar{\eta}$ has the same end values as η and

$$\begin{aligned} J(\xi) = J(\eta + b\bar{\eta}) &= J(\eta) + 2bJ(\eta, \bar{\eta}) + b^2 J(\bar{\eta}) \\ &= J(\eta) + 2bJ(\eta, \bar{\eta}). \end{aligned}$$

By a proper choice of b the last quantity can be made negative and the theorem is proved.

THEOREM 4:3. *Let* ξ_α $(\alpha = 1, \cdots, r_1)$ *be a maximal set of extremal arcs in* $\mathcal{E}^*$ *no proper linear combination of which is a transversal arc for* P. *Let* η_β $(\beta = 1, \cdots, r_2)$ *be a maximal set of arcs in* $\mathcal{E}$ *the end points of no linear combination of which can be joined by an extremal arc. Then* $r_1 = r_2$.

For if $r_2 > r_1$ there exist constants a_β such that $J(\xi_\alpha, \eta_\beta a_\beta) = 0$. By Corollary 1 of Theorem 4:1 the arc $\eta = a_\beta \eta_\beta$ can be joined by an extremal arc which is a contradiction. If $r_2 < r_1$ there exist constants b_α such that $J(b_\alpha \xi_\alpha, \eta_\beta) = 0$. It follows that $J(b_\alpha \xi_\alpha, \eta) = 0$ for every arc η in $\mathcal{E}$. Hence by Theorem 3:1 the arc $\xi = b_\alpha \xi_\alpha$ is a transversal arc contrary to our assumption. Therefore $r_1 = r_2$.

Concerning sub-problems we have several useful theorems one of which is the following:

THEOREM 4:4. *A necessary and sufficient condition that an arc* ξ *in* $\mathcal{E}$ *be expressible in the form*

$$(4:2) \qquad \xi = \xi_1 + \xi_2$$

where ξ_1 *is an arc in* $\mathcal{E}^\circ$ *and* ξ_2 *is a transversal arc for* P^* *is that* $J(\xi, \eta) = 0$ *for all extremal arcs* η *in* $\mathcal{E}^*$. *Moreover* $J(\xi) = J(\xi_1) + J(\xi_2)$.

The condition is necessary since the equations

$$J(\xi, \eta) = J(\xi_1 + \xi_2, \eta) = J(\xi_2, \eta)$$

hold for every extremal arc η in $\mathcal{E}^*$. By Theorem 3:1 it follows that $J(\xi_2, \eta) = 0$. We first show that the condition is sufficient when ξ is an extremal arc in $\mathcal{E}$ satisfying $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. In order to do this let ξ_α $(\alpha = 1, \cdots, r)$ be a maximal set of extremal arcs in $\mathcal{E}^\circ$ no proper linear combination of which satisfies the transversality conditions for P^*. Let η_γ $(\gamma = 1, \cdots, r)$ be a set of arcs in $\mathcal{E}^*$ such that the determinant $|J(\xi_\alpha, \eta_\gamma)|$ is different from zero. Such a set η_γ exists by virtue of Theorem 4:3. Hence there exist constants a_α such that

$$(4:3) \qquad J(\xi - a_\alpha \xi_\alpha, \eta_\gamma) = J(\xi, \eta_\gamma) - a_\alpha J(\xi_\alpha, \eta_\gamma) = 0.$$

We now show that $\xi - a_\alpha \xi_\alpha$ is a transversal arc for P^*. Let η be any arc in $\mathcal{E}^*$ and let b_γ be constants such that

$$(4:4) \qquad J(\xi_\alpha, \eta - b_\gamma \eta_\gamma) = J(\xi_\alpha, \eta) - b_\gamma J(\xi_\alpha, \eta_\gamma) = 0.$$

It follows from Corollary 1 of Theorem 4:1 that the arc $\eta - b_\gamma \eta_\gamma$ can be joined by an extremal arc $\bar{\eta}$. Since the arcs $\eta - b_\gamma \eta_\gamma$, $\bar{\eta}$ have the same end values and ξ is an extremal arc we have $J(\xi, \eta - b_\gamma \eta_\gamma) = J(\xi, \bar{\eta})$. Moreover $J(\xi, \bar{\eta}) = 0$ as a consequence of our assumption on ξ since $\bar{\eta}$ is an extremal arc in $\mathcal{E}^*$. Consequently $J(\xi, \eta - b_\gamma \eta_\gamma) = 0$. By the use of this equation and equations (4:4) it is found that

$$J(\xi - a_\alpha \xi_\alpha, \eta - b_\gamma \eta_\gamma) = J(\xi, \eta - b_\gamma \eta_\gamma) - a_\alpha J(\xi_\alpha, \eta - b_\gamma \eta_\gamma) = 0.$$

On the other hand

$$0 = J(\xi - a_\alpha \xi_\alpha, \eta - b_\gamma \eta_\gamma) = J(\xi - a_\alpha \xi_\alpha, \eta) - J(\xi - a_\alpha \xi_\alpha, \eta_\gamma) b_\gamma .$$

It follows from this relation and equations (4:3) that $J(\xi - a_\alpha \xi_\alpha, \eta) = 0$. Since η is any arc in $\mathcal{E}^*$ we see by the use of Theorem 3:1 that the arc $\xi - a_\alpha \xi_\alpha$ is a transversal arc for P^*. The arcs $\xi_1 = a_\alpha \xi_\alpha$, $\xi_2 = \xi - \xi_1$ are therefore related to ξ as described in the theorem. The criterion accordingly holds for an extremal arc ξ in $\mathcal{E}$.

Now suppose ξ is any arc in $\mathcal{E}$ satisfying $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. Since we have, in particular, that $J(\xi, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$ it follows from Corollary 2 of Theorem 4:1 that ξ is expressible in the form $\xi = \eta_0 + \eta_1$ where η_0 is in $\mathcal{E}^0$ and η_1 is an extremal arc in $\mathcal{E}$. Moreover since $J(\xi, \eta) = 0$ we have $J(\eta_1, \eta) = 0$ for all extremal arcs η in $\mathcal{E}^*$. Hence by the preceding argument η_1 is expressible in the form (4:2). It follows that ξ is also expressible in this form. The last statement of the theorem is clearly true since $J(\xi_1, \xi_2) = 0$.

THEOREM 4:5. *Let* ξ_α $(\alpha = 1, \cdots, s_1)$ *be a maximal set of arcs in* $\mathcal{E}$ *no proper linear combination of which is expressible in the form* (4:2). *Let* η_β $(\beta = 1, \cdots, s_2)$ *be a maximal set of extremal arcs in* $\mathcal{E}^*$ *no proper linear combination of which is a transversal arc for* P. *Then* $s_1 = s_2$.

For, if $s_1 > s_2$ there exist constants a_α such that $J(\xi_\alpha a_\alpha, \eta_\beta) = 0$. Since $J(\xi_\alpha a_\alpha, \eta) = 0$ for all transversal arcs η for P it follows by Theorem 4:4 that the arc $\xi = \xi_\alpha a_\alpha$ is expressible in the form (4:2) which is a contradiction. If

$s_1 < s_2$ there exist constants b_β such that $J(\xi_\alpha, \eta_\beta b_\beta) = 0$. The arc $\eta = b_\beta \eta_\beta$ then satisfies the condition $J(\eta, \xi) = 0$ for all arcs ξ in $\mathcal{E}$. Hence η is a transversal arc for P which is impossible. It follows that $s_1 = s_2$.

THEOREM 4:6. *Let* ξ_α ($\alpha = 1,\cdots, s_1$) *be a maximal set of extremal arcs in* $\mathcal{E}$ *no proper linear combination of which is expressible in the form* (4:2). *Let* η_β ($\beta = 1,\cdots,s_2$) *be a maximal set of extremal arcs in* $\mathcal{E}^*$ *no proper linear combination* η *of which is expressible in the form* $\eta = \eta_1 + \eta_2$ *where* η_1 *is an arc in* $\mathcal{E}^0$ *and* η_2 *is a transversal arc for* P. *Then* $s_1 = s_2$.

The number s_1 here is not necessarily the same as the number s_1 in Theorem 4:5.

As a generalization of Corollary 2 of Theorem 4:1 we have

THEOREM 4:7. *A necessary and sufficient condition that an arc* ξ *in* $\mathcal{E}$ *be expressible in the form*

$$\xi = \xi_1 + \xi_2 \tag{4:5}$$

where ξ_1 *is an arc in* $\mathcal{E}^*$ *and* ξ_2 *is a transversal arc for* P^* *is that* $J(\xi, \eta) = 0$ *for all transversal arcs* η *for* P^* *in* $\mathcal{E}^*$. *Moreover* $J(\xi) = J(\xi_1) + J(\xi_2)$.

The condition is necessary, for, as a consequence of Theorem 3:1 we have $J(\xi_1, \eta) = 0$, $J(\xi_2, \eta) = 0$ and hence $J(\xi, \eta) = 0$ for all transversal arcs η for P^* in $\mathcal{E}^*$. As in the proof of Theorem 4:4 we first show that the condition is sufficient for an extremal arc ξ in $\mathcal{E}$. To do this let η_α ($\alpha = 1,\cdots,s$) be a maximal set of extremal arcs in $\mathcal{E}^*$ such that the determinant $|J(\eta_\alpha, \eta_\beta)|$ is different from zero. It follows that if $\bar{\xi}$ is any extremal arc in $\mathcal{E}^*$ and $J(\eta_\alpha, \bar{\xi}) = 0$ then $J(\bar{\xi}) = 0$. Let η be any extremal arc in $\mathcal{E}^*$ and let a_β be a set of constants such that

$$(4{:}6)\qquad J(\zeta - a_\beta \zeta_\beta\,,\ \zeta_\alpha) = J(\zeta, \zeta_\alpha) - a_\beta J(\zeta_\beta\,,\ \zeta_\alpha) = 0.$$

The extremal arc $u = \zeta - a_\beta \zeta_\beta$ then satisfies $J(\zeta_\alpha, u) = 0$ and hence $J(u) = 0$, as was seen above. We can now show that $J(u, \bar\zeta) = 0$ for every extremal arc $\bar\zeta$ in $\mathcal{E}^*$. For, if $J(\zeta_\alpha, \bar\zeta) = 0$ then $J(\zeta_\alpha, u + \bar\zeta) = 0$ and hence $J(u + \bar\zeta) = 0$. Since

$$J(u + \bar\zeta) = J(u) + 2J(u, \bar\zeta) + J(\bar\zeta) = 2J(u, \bar\zeta),$$

it follows that $J(u, \bar\zeta) = 0$ as was to be proved. If $J(\zeta_\alpha, \bar\zeta) \neq 0$ then there exist constants b_β such that $J(\zeta_\alpha, \bar\zeta - b_\beta \zeta_\beta) = 0$ $(\beta = 1,\cdots,s)$. Hence $J(\bar\zeta - b_\beta \zeta_\beta) = 0$. Also we have the conditions $J(\zeta_\alpha, u + \bar\zeta - b_\beta \zeta_\beta) = 0$. It follows then that

$$0 = J(u + \bar\zeta - b_\beta \zeta_\beta) = 2J(u, \bar\zeta - b_\beta \zeta_\beta) = 2J(u, \bar\zeta).$$

As a consequence of Theorem 4:4 there exists an extremal arc ζ_0 in $\mathcal{E}^0$ such that the arc $u - \zeta_0$ is a transversal arc for P^*. Hence by our assumption on the extremal arc ξ we have $J(\xi, u - \zeta_0) = 0$. Clearly $J(\xi, \zeta_0) = 0$ and consequently $J(\xi, u) = 0$. Since there exist constants c_β such that

$$J(\xi - c_\beta \zeta_\beta\,,\ \zeta_\alpha) = J(\xi, \zeta_\alpha) - c_\beta J(\zeta_\beta\,,\ \zeta_\alpha) = 0,$$

it follows that

$$J(\xi - c_\beta \zeta_\beta\,,\ \zeta) = J(\xi - c_\beta \zeta_\beta\,,\ \zeta - a_\beta \zeta_\beta) = J(\xi - c_\beta \zeta_\beta\,, u),$$

the last equality holding because $\zeta - a_\beta \zeta_\beta = u$. On the other hand we have that

$$J(\xi - c_\beta \zeta_\beta\,, u) = J(\xi, u) - c_\beta J(\zeta_\beta\,, u) = 0$$

by virtue of equation (4:6) and $J(\xi, u) = 0$. From these relations it is seen that $J(\xi - c_\beta \zeta_\beta\,, \zeta) = 0$. Since ζ is any extremal arc in $\mathcal{E}^*$, it follows from Theorem 4:4 that there exists an extremal arc ξ_0 in $\mathcal{E}^0$ such that the arc $\xi - c_\beta \eta_\beta - \xi_0$ is a transversal arc for P^*. The arcs $\xi_1 = a_\beta \eta_\beta + \xi_0$ and $\xi_2 = \xi - \xi_1$ are related to ξ as described in the theorem. It follows that the condition of the theorem is sufficient for an extremal arc in $\mathcal{E}$.

Now suppose ξ is any arc in $\mathcal{E}$ and $J(\xi, \eta) = 0$ for every transversal arc η for P^* in $\mathcal{E}^*$. If the end points of ξ can be joined by an extremal arc η_1 then $\xi = \eta_0 + \eta_1$ where η_0 is an arc in $\mathcal{E}^0$. Also $J(\xi, \eta) = J(\eta_1, \eta) = 0$ for all transversal arcs η for P^* in $\mathcal{E}^*$. Hence by the preceding argument η_1 is expressible in the form (4:5). It follows that ξ is also expressible in this form. If the end points of ξ cannot be joined by an extremal arc let ξ_α $(\alpha = 1,\cdots,t)$ be a maximal set of extremal arcs in $\mathcal{E}^0$ no proper linear combination of which is a transversal arc for P^*. As a consequence of Theorem 4:3 there exists a set of arcs η_β $(\beta = 1,\cdots,t)$ in $\mathcal{E}^*$ such that the determinant $|J(\xi_\alpha, \eta_\beta)|$ is different from zero. Hence there exist constants d_β such that $J(\xi, \xi_\alpha) - d_\beta J(\eta_\beta, \xi_\alpha) = 0$. It follows that the arc $u_1 = \xi - d_\beta \eta_\beta$ can be joined by an extremal arc u_2 and $u_1 = u_0 + u_2$ where u_0 is an arc in $\mathcal{E}^0$. Also

$$J(u_2, \eta) = J(u_1, \eta) = J(\xi - d_\beta \eta_\beta, \eta) = J(\xi, \eta) = 0$$

for all transversal arcs η for P^* in $\mathcal{E}^*$. Again by the preceding argument u_2 is expressible in the form $u_2 = v_1 + v_2$, where v_1 is in $\mathcal{E}^*$ and v_2 is a transversal arc for P. It follows that $\xi = \xi_1 + \xi_2$ where $\xi_1 = u_0 + v_1 + d_\beta \eta_\beta$ is in $\mathcal{E}^*$ and $\xi_2 = v_2$ is the transversal arc for P^*, as was to be proved. The last statement of the theorem follows readily since $J(\xi_1, \xi_2) = 0$.

THEOREM 4:8. *Let* ξ_α $(\alpha = 1,\cdots,s_1)$ *be a maximal set of arcs in* $\mathcal{E}$ *no proper linear combination of which is expressible in the form* (4:4). *Let* η_β $(\beta = 1,\cdots,s_2)$ *be a maximal set of transversal arcs for* P^* *in* $\mathcal{E}^*$ *no proper linear combination of which is a transversal arc for* P. *Then* $s_1 = s_2$.

5. *The index of a problem with respect to a sub-problem.* Let P be a problem and P^* a sub-problem of P of the type studied in Section 4. Denote the indices of P and P^* by k and k^*, respectively, where by the index of P is meant the index of $J(\eta)$ on $\mathcal{E}$. Similarly the index of P^* is the index of $J(\eta)$ on $\mathcal{E}^*$. The difference $q^* = k - k^*$ will be called the *index of* P *with respect to* P^*. In special instances Birkhoff and Hestenes (I) and Morse (VIII) have called a number equivalent to the difference q^* the order of concavity. In the case of periodic extremals the index of repletion as defined by Morse and Pitcher (X) may be interpreted as the index of a problem with respect to a sub-problem.

LEMMA 5:1. *Let* η_β $(\beta = 1,\cdots,g)$ *be a maximal set of extremal arcs in* $\mathcal{E}$ *every proper linear combination* η *of which has* $J(\eta) < 0$. *Let* η_γ $(\gamma = g+1, \cdots, g+h)$ *be a maximal set of extremal arcs in* $\mathcal{E}^0$ *no proper linear combination of which is a transversal arc for* P. *Then the index* q^0 *of* P *with respect to* P^0 *is given by the formula* $q^0 = g + h$.

In order to prove this we show that $k = k^0 + g + h$, where k and k^0 are the indices of P and P^0, respectively. Let ξ_ρ $(\rho = 1,\cdots,k^0)$ be a maximal set of arcs belonging to $\mathcal{E}^0$ every proper linear combination η of which has $J(\eta) < 0$. It is now sufficient to show that the conditions

$$(5{:}1)\qquad \begin{aligned} &J(\xi_\rho, \eta) = 0 &&(\rho = 1,\cdots,k^0),\\ &J(\eta_\alpha, \eta) = 0 &&(\alpha = 1,\cdots, g + h)\end{aligned}$$

form a minimal set for P. Since every proper linear combination η of these arcs has $J(\eta) \leqq 0$ but no proper linear combination is a transversal arc for P, it follows as a consequence of Theorem 3:5 that none of the conditions (5:1) can be discarded. In order to show that there are enough conditions let η be any arc in $\mathcal{E}$ satisfying the conditions (5:1). Since η satisfies the conditions $J(\eta_\gamma, \eta) = 0$ ($\gamma = g + 1,\cdots,g + h$), it follows from Corollary 1 of Theorem 4:1 that the end points of η can be joined by an extremal arc, that is, that η is expressible in the form $\eta = \eta_1 + \eta_2$ where η_1 is an arc in $\mathcal{E}^0$ and η_2 is an extremal arc in $\mathcal{E}$. By virtue of the fact that η_2 is an extremal arc $J(\xi_\rho, \eta_2) = 0$. Moreover, since η_2 has the same end values as η, we have $J(\eta_\alpha, \eta_2) = 0$. Hence η_2 satisfies the conditions (5:1). It follows then that η_1 also satisfies these conditions. As a consequence of our choice of the ξ_ρ and the η_α we have $J(\eta_1) \geqq 0$ and $J(\eta_2) \geqq 0$. Hence $J(\eta) \geqq 0$, and the theorem follows immediately.

By means of this lemma we can prove the following significant result:

THEOREM 5:1. *The index* q^* *of* P *with respect to* P^* *is equal to the number of extremal arcs* η_δ ($\delta = 1,\cdots,s$) *in a maximal set of arcs in* $\mathcal{E}$ *every linear combination* η *of which is a transversal arc for* P^* *and has* $J(\eta) \leqq 0$ *but no proper linear combination of which is a transversal arc for* P.

To prove this it is sufficient to show that $k = k^* + s$ where k and k^* are the indices of P and P^*, respectively. Consider now the conditions

$$(5{:}2)\qquad \begin{aligned} &J(\xi_\rho, \eta) = 0 &&(\rho = 1,\cdots,k^*),\\ &J(\eta_\sigma, \eta) = 0 &&(\sigma = 1,\cdots,s), \end{aligned}$$

the first k^* of which form a minimal set for P^*. By Lemma 5:1 we may replace these k^* conditions by a set of the form (5:1) defined now for the problem P^*. Assume that this replacement has been made. We now wish to show that the k^*+s conditions thus defined form a minimal set for P. As a consequence of Theorem 3:1 applied to the problem P^* the extremal arcs η_σ satisfy the conditions $J(\xi_\rho, \eta_\sigma) = 0$. With the help of this fact we have that every linear combination $\bar{\eta}$ of the arcs ξ_ρ, η_σ has $J(\bar{\eta}) \leqq 0$. Since no proper linear combination of these arcs is a transversal arc for P, it now follows that none of the conditions (5:2) can be dropped. In order to show that there are enough conditions let η be any arc in $\mathcal{E}$ satisfying the conditions (5:2). Clearly $J(u, \eta) = 0$ for all transversal arcs u for P^* in $\mathcal{E}^*$ every linear combination of which is a transversal arc for P. Since $J(\eta_\sigma, \eta) = 0$, we have that η satisfies the condition $J(\xi, \eta) = 0$ for all transversal arcs ξ for P^* in $\mathcal{E}^*$. Hence by Theorem 4:7 the arc η is expressible in the form $\eta = \eta_1 + \eta_2$ where η_1 is an arc in $\mathcal{E}^*$ and η_2 is a transversal arc for P^*. As a consequence of Theorem 3:1 we have $J(\eta_\sigma, \eta_1) = 0$, $J(\xi_\rho, \eta_2) = 0$. It follows then that $J(\eta_\sigma, \eta_2) = 0$, $J(\xi_\rho, \eta_1) = 0$. By virtue of our choice of the arcs ξ_ρ, η_σ we have $J(\eta_1) \geqq 0$, $J(\eta_2) \geqq 0$. Since $J(\eta) = J(\eta_1) + J(\eta_2)$ it follows that $J(\eta) \geqq 0$.

COROLLARY 1. *The index* q^0 *of* P *with respect to the fixed end point problem* P^0 *is equal to the number of extremal arcs* η_α $(\alpha = 1,\cdots,q^0)$ *in a maximal set of arcs in* $\mathcal{E}$ *every linear combination* η *of which has* $J(\eta) \leqq 0$ *but no proper linear* *combination of which is a transversal arc for* P.

COROLLARY 2. *A necessary and sufficient condition that the index of* P *be equal to zero is that the index of* P^0 *be equal to zero and that the inequality* $J(\eta) \geqq 0$ *hold for all extremal arcs* η *in* $\mathcal{E}$, *the equality holding only in case* η *is a transversal arc for* P.

COROLLARY 3. *If there exist no extremal arcs in* $\mathcal{E}^0$ *then the index* q^0 *of* P *with respect to* P^0 *is equal to the number of extremal arcs* η_α $(\alpha = 1,\cdots,q^0)$ *in a maximal set of arcs in* $\mathcal{E}$ *every proper linear combination* η *of which has* $J(\eta) < 0$.

Consider again a problem P and a sub-problem P^* of the type already studied. We then have

LEMMA 5:2. *Let* p, p^* *be the number of linearly independent end conditions for the problems* P, P^*, *respectively. Then the index* q^* *of* P *with respect to* P^* *satisfies the inequality*

$$q^* \leqq p^* - p.$$

For, let ξ_α $(\alpha = 1,\cdots,k^*)$ be a set of arcs in $\mathcal{E}^*$ such that the conditions $J(\xi_\alpha, \eta) = 0$ $(\alpha = 1,\cdots,k^*)$ form a proper minimal set for P^*. Since P^* is a sub-problem of P, any arc satisfying the end conditions for P^* also satisfies the end conditions for P. Let η_γ $(\gamma = 1,\cdots,q^*)$ be a maximal set of arcs in $\mathcal{E}$ satisfying the conditions $J(\xi_\alpha, \eta) = 0$ every proper linear combination η of which has $J(\eta) < 0$. Since no proper linear combination of the arcs η_γ is in $\mathcal{E}^*$ and therefore not in $\mathcal{E}^0$, it follows that no two of the arcs η_γ have the same end values. Hence $q^* \leqq (2n - p) - (2n - p^*) = p^* - p$.

COROLLARY. *The index* q^0 *of* P *with respect to* P^0 *satisfies the inequality*

$$q^0 \leqq 2n - p.$$

For, in this case the number corresponding to p^* of Lemma 5:2 is equal to 2n.

It can be shown that the definition of the index of a problem with respect to a sub-problem as given by Birkhoff and Hestenes (I, 225) is not properly stated. If the extremal arcs η_α ($\alpha = 1,\cdots,h$) used by Birkhoff and Hestenes in the definition are required to be transversal arcs for the problem P^*, then their definition is equivalent to the one here given. The index of a problem with respect to a sub-problem was called, by Birkhoff and Hestenes, the order of concavity of a problem with respect to a sub-problem.

We now consider another type of sub-problem which will be of use in the following section. Let P be a problem of the type defined in Section 4 and let P^* be a sub-problem obtained from P by adding the conditions

$$\int_{x_1}^{x_2} \eta_{i\alpha}\,\eta_i dx = 0 \qquad (\alpha = 1,\cdots,s), \tag{5:3}$$

where the arcs η_α ($\alpha = 1,\cdots,s$) form a maximal set of linearly independent transversal arcs for P. The class of arcs in $\mathcal{E}$ which satisfy the conditions (5:3) shall be denoted by $\mathcal{E}^*$. We then have the following result:

THEOREM 5:2. *The index of* P^*, *where* P^* *is the sub-problem defined above, is equal to the index of* P. *Moreover there exist no transversal arcs for* P^* *in* $\mathcal{E}^*$ *except the arc* $(\eta) \equiv (0)$.

In order to prove the first statement of the theorem suppose the index of P is k and let the conditions $J(\zeta_\gamma, \eta) = 0$ ($\gamma = 1,\cdots,k$) form a proper minimal set for P. If the arcs ζ_γ are in $\mathcal{E}^*$ the theorem is obviously true for then the conditions $J(\zeta_\gamma, \eta) = 0$ also form a proper minimal set for P^*. If the arcs ζ_γ are not in $\mathcal{E}^*$ then there exist constants $a_{\alpha\gamma}$ such that the relations

$$\int_{x_1}^{x_2} \zeta_\beta (a_{\alpha\gamma} \zeta_\alpha + \xi_\gamma) dx = 0 \qquad (\alpha, \beta = 1, \cdots, s;\ \gamma = 1, \cdots, k)$$

hold, that is, the arcs $a_{\alpha\gamma} \zeta_\alpha + \xi_\gamma$ are in $\mathcal{E}^*$. Let $u_\gamma = a_{\alpha\gamma}\zeta_\alpha$, then it can be shown that the conditions $J(\xi_\gamma + u_\gamma, \eta) = 0$ form a minimal set for P^*. For, if η is any arc in $\mathcal{E}$, then $J(u_\gamma, \eta) = 0$ since the arcs ζ_α and hence u_γ are transversal arcs for P. We then have

$$J(\xi_\gamma + u_\gamma, \eta) = J(\xi_\gamma, \eta) + J(u_\gamma, \eta) = J(\xi_\gamma, \eta).$$

Since the conditions $J(\xi_\gamma, \eta) = 0$ form a proper minimal set for P, it follows that any arc η in $\mathcal{E}$ satisfying these conditions has $J(\eta) \geqq 0$. Hence any arc η in $\mathcal{E}^*$ satisfying $J(\xi_\gamma + u_\gamma, \eta) = 0$ has $J(\eta) \geqq 0$. Furthermore any arc of the form $\eta = \xi + u$ where $\xi = \xi_\gamma b_\gamma$, $u = u_\gamma b_\gamma$ gives $J(\eta)$ a negative value, for, $J(\xi, u) = J(u) = 0$, as is readily seen, and hence

$$J(\xi + u) = J(\xi) + 2J(\xi, u) + J(u) = J(\xi).$$

The functional $J(\xi)$ is less than zero by choice of the ξ_γ.
In order to prove the second statement of the theorem we first note that the Euler equations for P^* are of the form

$$\Omega_{\eta_1} - \frac{d}{dx} \Omega_{\eta_1'} = \mu_\alpha \zeta_\alpha \qquad (\alpha = 1, \cdots, s)$$

where the multipliers μ_α are constants (II, 699). An extremal arc then is a set η_1, ξ_1, μ_α satisfying these equations. If the arc η is a transversal arc for P^* in $\mathcal{E}$ then by an integration by parts it is seen that

$$0 = J(\zeta_\alpha, \eta) = -\mu_\beta \int_{x_1}^{x_2} \eta_{1\alpha} \zeta_{1\beta} dx \quad (\alpha, \beta = 1, \cdots, s).$$

Since the determinant $\left|\int_{x_1}^{x_2} \eta_{i\alpha}\,\eta_{i\beta}\,dx\right|$ is different from zero it follows that $\mu_\beta = 0$ $(\beta = 1,\cdots,s)$. Hence the transversal arcs for P^* in $\mathcal{E}$ are transversal arcs for P. Clearly the only transversal arcs in $\mathcal{E}$ satisfying the conditions (5:3), that is, the only transversal arc for P^* in $\mathcal{E}^*$ is $(\eta) \equiv (0)$.

6. The one variable end point problem and focal intervals. In this section we shall consider the one variable end point problem P determined by the functional $J(\eta)$ and a set of arcs $\mathcal{E}$ consisting of those arcs in $\mathcal{D}$ satisfying the end conditions

$$(6:1) \qquad \Psi_\mu = a_{\mu i}\,\eta_i(x_1) = 0, \qquad \eta_i(x_2) = 0 \qquad (\mu = 1,\cdots,p \leqq n).$$

The matrix $\|a_{\mu i}\|$ is assumed to be of rank p. The functional $J(\eta)$ then takes the form

$$(6:2) \qquad J(\eta) = 2q(\eta_1) + \int_{x_1}^{x_2} 2\omega(\eta,\eta')dx,$$

and the transversality conditions are given by

$$(6:3) \qquad T_i(\eta,\gamma,\lambda) = q_{\eta_{i1}} + \gamma_\mu a_{\mu i} - \zeta_i(x_1) = 0.$$

A transversal arc for P which satisfies the conditions $\Psi_\mu = 0$ will be called a focal arc.

The problem P determined by $J(\eta)$ in the form (6:2) is equivalent to a problem determined by $J(\eta)$ in the form

$$(6:4) \qquad J(\eta,w) = b_{h\gamma}w_h w_\gamma + \int_{x_1}^{x_2} 2\omega(\eta,\eta')dx \qquad (h,\gamma = 1,\cdots,r)$$

where the matrix $\|b_{h\gamma}\|$ is symmetric and the numbers w_h are constants. The end conditions are of the form

$$\eta_i(x_1) = c_{ih}w_h, \qquad \eta_i(x_2) = 0 \qquad (h = 1,\cdots,r)$$

where the matrix $\|c_{ih}\|$ has maximum rank r, and the transversality conditions are

$$\zeta_i(x_1)c_{ih} = b_{h\gamma}w_\gamma.$$

By adding the conditions $w_h' = \eta'_{n+h} = 0$ it is seen that the problem formulated in this way is equivalent to a problem of the type P in (n+r+1)-dimensional space. In general we shall consider $J(\eta)$ as given in the form (6:2). We shall use the form (6:4) whenever it seems more convenient to do so.

An interval x'x", not containing x_1, will be called a *focal interval* of x_1, if there exists a focal arc η which is identically zero on x'x", is not identically zero on an extension of x'x" and is such that there exists no focal arc ζ having $\zeta \equiv \eta$ on an extension of x'x" to the left of x' and $\zeta \equiv 0$ on an extension of x'x" to the right of x" (cf. Reid (XII, 277)). A focal arc η with the above properties will be said to correspond to the focal interval x'x". Any point on a focal interval x'x" will be called a *focal point*; the point x' being called, in particular, a *primary focal point*, and the point x" a *secondary focal point*. Frequently a focal interval may consist of a single point. This is always the case when the problem is normal on every sub-interval (II, 725). The *order of a focal interval* x'x" is defined to be the number of linearly independent focal arcs in a maximal set, every proper linear combination of which corresponds to the focal interval x'x" in the sense described above. If the conditions $\Psi_\mu = 0$ are of the form $\eta_i(x_1) = 0$ then the problem P reduces to the fixed end point problem P^0 and focal intervals (points) are called *conjugate intervals* (points).

It may be noted that in the definition of a focal interval x'x" given above the words "left" and "right" may be interchanged. To prove this we need only observe that if there exists a focal arc $\bar{\xi}$ having $\bar{\xi} \equiv \eta$ on an extension of $x'x''$ to the right of x'' and $\bar{\xi} \equiv 0$ on an extension of $x'x''$ to the left of x', then the arc $\xi = \eta - \bar{\xi}$ is a focal arc having $\xi = \eta$ on an extension of $x'x''$ to the left of x' and $\xi \equiv 0$ on an extension of $x'x''$ to the right of x'' and conversely.

We shall assume throughout this section that the strengthened Clebsch condition holds on the interval x_1x_2. We then have the following well-known result (IV).

THEOREM 6:1. *The inequality $J(\eta) > 0$ holds for every arc η in $\mathcal{E}$, that is, the index of P is equal to zero if and only if there exist no focal points of x_1 on x_1x_2.*

A more general theorem is the following one:

THEOREM 6:2. *The inequality $J(\eta) \geqq 0$ holds for every arc in $\mathcal{E}$, that is, the index of P is equal to zero if and only if there exist no focal intervals of $x = x_1$ interior to x_1x_2. Moreover, the number of linearly independent arcs in $\mathcal{E}$ having $J(\eta) = 0$ is equal to the number of focal intervals containing x_2.*

For, if there were a focal interval $x'x''$ of x_1 interior to x_1x_2 there would exist an arc in $\mathcal{E}$, namely, an arc which is a transversal arc on x_1x'' and identically zero on $x''x_2$ having $J(\eta) = 0$ but which is not a transversal arc on x_1x_2. Hence the index of P on $\mathcal{E}$ could not be equal to zero. In order to prove the converse suppose there exist no focal intervals interior to x_1x_2. If in addition there are no focal intervals containing x_2, then $J(\eta) > 0$ by Theorem 6:1. Now suppose there are focal intervals containing x_2 and let x' be the first primary focal point. Consider first the case where x' is the only primary focal point. Let ζ_α $(\alpha = 1, \cdots, t)$ be a maximal set of linearly independent focal arcs vanishing on $x'x_2$ and define P^* to be the sub-problem obtained by adding the conditions

$$(6{:}5) \qquad \int_{x_1}^{x_2} \zeta_{1\alpha}\,\zeta_1 dx = 0 \qquad (\alpha = 1,\cdots,t).$$

The problem P^* is then a sub-problem of the type discussed in the last part of Section 5. By Theorem 5:2 the only transversal arc in $\mathcal{E}$ satisfying the conditions (6:5) is $(\zeta) \equiv (0)$. Hence there exist no focal intervals of x_1 for P^* which contain x_2. Since $\zeta_{1\alpha} \equiv 0$ on $x'x_2$, the point x_2 can be replaced in the above argument by any point on $x' \leqq x \leqq x_2$ without altering the conditions (6:5). It follows that no point of $x'x_2$ can belong to a focal interval of x_1 for P^*. Moreover, there can be no focal point of x_1 for P^* on $x_1 < x < x'$. For, if there were such a point, say x_3, then there would exist a focal arc ζ for P^* having $\zeta_1(x_3) = 0$ and $J(\zeta) = 0$ on x_1x_3. But since there are no focal points of x_1 for P on x_1x_3, we have by Theorem 6:1 that $J(\zeta) > 0$ on this interval. It follows again from Theorem 6:1 that the index of P^* is equal to zero. Hence by Theorem 5:2 the index of P is also equal to zero, that is, $J(\zeta) \geqq 0$ for every arc ζ in $\mathcal{E}$.

Now consider the case where there are other primary focal points on $x'x_2$ besides x' and let x'' be the first of these. As a consequence of the case already considered the index of $J(\zeta)$ is equal to zero on x_1x''. Let P' denote the sub-problem of P obtained by adding the conditions $\int_{x_1}^{x_2} \zeta_{1\beta}\,\zeta_1\, dx = 0$ where the arcs ζ_β $(\beta = 1,\cdots,t')$ form a maximal set of linearly independent focal arcs vanishing at x'. We next obtain a sub-problem P'' of P' by adding conditions $\int_{x_1}^{x_2} \zeta_{1\gamma}\,\zeta_1 dx = 0$ where the arcs ζ_γ $(\gamma = t'+1,\cdots,t'+t'')$ form a maximal set of linearly independent arcs vanishing at x''. We then show by the methods used before that the index of P'' is zero on x_1x''' where x''' is the third primary focal point of x_1. It follows that the index of P is also zero on this interval. We can continue to form sub-problems in this way. Since, in this case, the number of primary focal points of x_1 on x_1x_2 is at most n it follows that we finally obtain a sub-problem the index of which is equal to zero on the entire interval x_1x_2. The index of P then is also zero on x_1x_2. In fact the index remains equal to zero to the first secondary focal point. The last statement of the theorem is clearly true.

Consider now the expression

$$(6{:}6) \qquad J(\eta, a{:}x) = \eta_{i\sigma}(x)\,\zeta_{i\tau}(x)a_\sigma a_\tau + \int_x^{x_2} 2\omega(\eta, \eta')dx$$

which is a functional of the form (6:4). The arcs η_σ are taken to be a maximal set of focal arcs no linear combination of which vanishes at the point x. The range of σ and τ depends then upon x. This functional determines a problem P_x having the end conditions

$$(6{:}7) \qquad \eta_i(x) = \eta_{i\sigma}(x)a_\sigma, \qquad \eta_i(x_2) = 0$$

and the transversality conditions

$$(6{:}8) \qquad \zeta_i(x)\,\eta_{i\sigma}(x) = \eta_{i\sigma}(x)\,\zeta_{i\tau}(x)a_\tau .$$

As a consequence of these conditions we have the result:

LEMMA 6:1. An interval interior to xx_2 is a focal interval for P_x if and only if it is a focal interval for P.

For, suppose a focal interval interior to x_1x_2 is determined for the problem P by a focal arc η. This arc η then satisfies the first set of conditions (6:7). Hence the focal interval thus determined is also a focal interval for P_x. Conversely, an arc η determining a focal interval for P_x must satisfy the first of conditions (6:7) and the conditions (6:8). It follows that η also determines a focal interval for P.

Let x' be any point on x_1x_2. We then have the following useful lemmas.

LEMMA 6:2. *If* η *is any arc in* $\mathcal{E}$ *and if* $J(\eta, \zeta) = 0$ *for all arcs* ζ *in* $\mathcal{E}$ *which are identically zero on* $x'x_2$ *then* $J(\eta) = J(\eta, a:x')$ *where* $J(\eta, a:x')$ *is defined in* (6:6).

For, by Theorem 3:1 the arc η is a transversal arc on x_1x'. Hence by an integration by parts

$$J(\eta) = \eta_1(x')\zeta_1(x') + \int_{x'}^{x_2} 2\omega(\eta, \eta')dx.$$

If η_σ $(\sigma = 1,\cdots,r')$ is a maximal set of linearly independent focal arcs not vanishing at x' then

$$J(\eta) = \eta_{1\sigma}(x')\zeta_{1\tau}(x')a_\sigma a_\tau + \int_{x'}^{x_2} 2\omega(\eta, \eta')dx = J(\eta, a:x') \qquad (\sigma, \tau = 1,\cdots,t').$$

LEMMA 6:3. *If* η *is any arc in* $\mathcal{E}$ *and if* $J(\eta, \zeta) = 0$ *for all arcs which are focal arcs on* x_1x' *and identically zero on* $x'x_2$ *then*

$$J(\eta) = J(\eta, a:x') + J(\eta_1),$$

where η_1 *is an arc in* $\mathcal{E}$ *identically zero on* $x'x_2$.

For, by Theorem 4:7 the arc η on the interval x_1x' is expressible in the form $\eta = \eta_1 + \eta_2$ where η_1 is an arc in $\mathcal{E}$ vanishing at x_1 and η_2 is a focal arc. Hence on the interval x_1x_2

$$J(\eta) = J(\eta_1) + \eta_{12}(x')\zeta_{12}(x') + \int_{x'}^{x_2} 2\omega(\eta, \eta')dx.$$

As in the proof of Lemma 6:2 it can be shown that the sum of the last two terms is expressible in the form $J(\eta, a:x')$ with a suitable set of focal arcs η_σ and constants a_σ.

COROLLARY. *If* η *is any arc in* $\mathcal{E}$ *satisfying the conditions of Lemma* 6:3 *and if there exist no focal intervals of* x_1 *interior to* x_1x', *then* $J(\eta) \geqq J(\eta, a:x')$.

This follows from the fact that $J(\eta_1) \geqq 0$ by Theorem 6:2.

It may be noted that Lemmas 6:2 and 6:3 still hold if the arc η is restricted only to the class ϑ instead of to $\mathcal{E}$ provided we choose the arcs $\eta_{1\sigma}$ occurring in the form (6:6) to be a maximal set of linearly independent transversal arcs not vanishing at the point x'.

When the Clebsch condition holds in the strengthened form the index of P (and hence also of P_x) is finite, as will be seen in Theorem 7:4 below. Assuming for the present the truth of this fact we can prove the following theorem:

THEOREM 6:3. *The index of* $J(\eta)$ *on* $\mathcal{E}$ *is equal to the number of focal intervals of* $x = x_1$ *interior to* x_1x_2 *where a focal interval is counted a number of times equal to its order*.

In order to prove this let x' be the first primary focal point of x_1. Denote by ζ_α $(\alpha = 1, \cdots, t')$ a maximal set of arcs in $\mathcal{E}$ which are identically zero on $x'x_2$, are focal arcs on x_1x' and are such that no proper linear combination of these arcs are transversal arcs. It is clear that t' is equal to the number of focal intervals having x' as their initial point, each focal interval being counted a number of times equal to its order. Consider an arc η in $\mathcal{E}$ satisfying the conditions $J(\zeta_\alpha, \eta) = 0$. By Lemma 6:3 the value of $J(\eta)$ is equal to the sum $J(\eta) = J(\eta, a:x') + J(\eta_1)$ where η_1 is a suitably chosen arc in $\mathcal{E}$, identically zero on $x'x_2$ and where $J(\eta, a:x')$ is an expression of the type (6:6), the range of σ, τ being at most $n - t'$ in this case. The focal intervals for the problem $P_{x'}$

determined by the functional $J(\eta, a:x')$ as described above are the focal intervals for P on $x'x_2$ which do not have x' as their initial point. Moreover the difference between the indices of P and $P_{x'}$ is equal to t'. For, let the conditions $J(u_\gamma, \eta) = 0$ $(\gamma = 1, \cdots, s')$ form a minimal set for $P_{x'}$ and let $\mathcal{E}_{x'}$ denote the class of arcs associated with $P_{x'}$. It is clear from the way in which $P_{x'}$ was constructed that the arcs in $\mathcal{E}_{x'}$ are equivalent to arcs in $\mathcal{E}$ which are focal arcs on the interval x_1x'. Hence we may consider the arcs u_γ as being arcs in $\mathcal{E}$, which are focal arcs on the interval x_1x'. We now show that the conditions $J(u_\gamma, \eta) = 0$, $J(\xi_\alpha, \eta) = 0$ form a minimal set for P. Clearly none of these conditions can be discarded. Now let ξ be any arc in $\mathcal{E}$ satisfying these conditions. We wish to show that $J(\xi) \geqq 0$. By Lemma 6:3 we have $J(\xi) = J(\xi, a:x') + J(\xi_1)$, where ξ_1 is an arc in $\mathcal{E}$ identically zero on $x'x_2$. Since x' is the first primary focal point of x_1 it follows from the corollary to Lemma 6:3 that $J(\xi_1) \geqq 0$. Also $J(\xi, a:x') \geqq 0$ for, otherwise the conditions $J(u_\gamma, \eta) = 0$ could not form a minimal set for $P_{x'}$. Hence $J(\xi) \geqq 0$.

Now suppose x" is the first primary focal point of x' for $P_{x'}$ and construct a problem $P_{x''}$ determined by a functional $J(\eta, a:x'')$. By an argument of the type already given it follows that the difference between the indices of $P_{x'}$ and $P_{x''}$ is equal to the number of focal intervals for x' which have x" as their initial point. The difference between the indices of P and $P_{x''}$ is then the number of focal intervals of x_1 interior to x_1x_2 and having x' and x" as their initial points. We continue to form problems of this type at each primary focal point. At the r-th step the index of the problem P_{x^r} which corresponds to the primary focal point x^r is less than that of the problem defined at the preceding step by an amount equal to the number of focal intervals interior to x_1x_2 and having x^r as their initial point. Since the index of P is finite one obtains after a finite number of steps a problem P_{x^m} whose index is zero. The index of P is accordingly equal to the number of focal intervals interior to x_1x_2 and having x', x'', $\cdots$, x^m as their initial points. Since the index of P_{x^m} is zero, there are no focal intervals of x_1 or of x^m interior to x^mx_2 by virtue of Lemma 6:1 and Theorem 6:2. Thus we have accounted for all focal intervals of x_1 on the interior of x_1x_2. This proves the theorem.

Let P and P^0 be problems as defined in Section 4. Since P^0 is a special case of the one variable end point problem, we have

COROLLARY 1. *The index of* P^0 *is equal to the number of conjugate intervals of* x_1 *interior to* x_1x_2.

COROLLARY 2. *The index of* P *is equal to the number of conjugate intervals of* x_1 *interior to* x_1x_2 *plus the index of* P *with respect to* P^0.

LEMMA 6:4. *Given a conjugate system* η_{ik}, ζ_{ik} *there exists a functional* $J(\eta)$ *of the form* (6:1) *and a set of end conditions of the form* (6:2) *for which the extremals* η_i, ζ_i *satisfying the end conditions* $\Psi_\mu = 0$ *and the transversality conditions* (6:3) *are of the form* $\eta_i = \eta_{ik}a_k$, $\zeta_i = \zeta_{ik}a_k$ *where the* a's *are constants not all zero.*

In order to prove this suppose n - p is the rank of the matrix $\|\eta_{ik}(x_1)\|$ and let the functions η_{ik} be chosen so that $\eta_{i\mu}(x_1) = 0$ ($\mu = 1,\cdots,p$). The end conditions are then chosen to be $\Psi_\mu = a_{\mu i}\eta_i(x_1) = 0$ where $a_{\mu i} = \zeta_{i\mu}(x_1)$, and the quadratic form 2q in (6:1) is taken to be $2q = \frac{1}{2}\left[\zeta_{ki}(x_1) + \zeta_{ik}(x_1)\right]\eta_{i1}\eta_{k1}$. The problem formed in this way satisfies the properties of the theorem as one can readily verify.

An interval $x'x''$ is said to be a *focal interval of a conjugate system* η_{ik}, ζ_{ik} if there exists an arc of the form $\eta_i = \eta_{ik}a_k$ identically zero on $x'x''$ but not identically zero on an extension of $x'x''$ and if there exists no arc of the form $\xi_i = \eta_{ik}b_k$ having $\xi \equiv \eta$ on an extension of $x'x''$ to the left of x' and $\xi \equiv 0$ on an extension of $x'x''$ to the right of x''. As in the definition of a focal interval given earlier in this section, the words "left" and "right" may also be interchanged here. A focal interval of a conjugate system on an interval x_1x_2 may also be said to be a focal interval of a one variable end point of the problem determined by the system. Let η_{ik}, u_{ik} be the extremal arcs associated with two conjugate systems and let P_1, P_2 be the problems determined by these systems on the interval x_1x_2. For convenience suppose the variable end point for P_1 is x_1 and the variable end point for P_2 is x_2. If we let P^0 denote the fixed end point problem then we have the following result.

LEMMA 6:5. *Let* t_1 *denote the number of focal intervals of* x_1 *and* t_2 *the number of focal intervals of* x_2 *interior to the interval* x_1x_2 *for the problems* P_1 *and* P_2 *respectively, where* P_1 *and* P_2 *are defined as above. Let* q_1, q_2 *denote respectively the indices of* P_1, P_2 *with respect to* P^0. *Then*

$$t_1 - t_2 = q_1 - q_2.$$

This follows from Corollary 2 of Theorem 6:3 since the number of conjugate intervals of x_1 is the same as the number of conjugate intervals of x_2 interior to x_1x_2.

We now consider an example of a one variable end point problem P in a 4-dimensional space having focal intervals $x'x''$, $t't''$ lying interior to x_1x_2 with $x' < t' \leqq x''$. Let $u(x)$ be an arc of class C'' such that $u(x_1) = 0$ and $u(x) \equiv 0$ on $x'x''$ but not on an extension of $x'x''$. Similarly let $v(x)$ be an arc of class C'' such that $v(x) \equiv 0$ on x_1x', and $t't''$ but not on an extension of $t't''$. Let $h_1 = u'' + u$, $h_2 = v'' + v$ and

$$b_1 = \int_{x_1}^{x'} uh_1dx = -\int_{x_1}^{x'} (u'^2 - u^2)dx, \qquad b_2 = \int_{x_1}^{t'} vh_2dx = -\int_{x_1}^{t'} (v'^2 - v^2)dx.$$

The problem P under consideration is determined by the functional

$$J(\eta, w) = b_1w_1^2 + b_2w_2^2 + \int_{x_1}^{x_2} (\eta_1'^2 - \eta_1^2)dx,$$

the differential equations

$$\eta_2' = \eta_1h_1, \qquad \eta_3' = \eta_1h_2$$

and the end conditions

$$\eta_1(x_1) = 0, \quad \eta_2(x_1) = -b_1w_1, \quad \eta_3(x_1) = -b_2w_2, \quad \eta_i(x_2) = 0$$
$$(i = 1, 2, 3),$$

where w_1, w_2 are constants. The Euler equations and transversality conditions associated with the problem P are, respectively,

$$\eta_1'' + \eta_1 + \lambda_1h_1 + \lambda_2h_2 = 0, \qquad \lambda_1' = 0, \qquad \lambda_2' = 0,$$

and

$$\zeta_2(x_1) = -w_1, \qquad \zeta_3(x_1) = -w_2.$$

It is readily verified that the arc η defined by

$$(6{:}9) \quad \eta_1 = u, \quad \eta_2 = \int_{x_1}^{x} uh_1dx - b_1w_1, \quad \eta_3 = \int_{x_1}^{x} uh_2dx - b_2w_2$$

with multipliers $\lambda_1 = -1$, $\lambda_2 = 0$ and constants $w_1 = 1$, $w_2 = 0$ is a focal arc for P and is identically zero on $x'x''$. Similarly the arc η defined by

$$\eta_1 = v, \quad \eta_2 = \int_{x_1}^{x} vh_1dx - b_1w_1, \quad \eta_3 = \int_{x_1}^{x} vh_2dx - b_2w_2$$

with multipliers $\lambda_1 = 0$, $\lambda_2 = -1$ and constants $w_1 = 0$, $w_2 = 1$ is also a focal arc for P and is identically zero on $t't''$. If $t' > x''$ we choose the function $v(x)$ so that $\int_{x''}^{t'} vh_1dx = 0$. It may be noted that if $x_1 = 0$ and $x' > \pi$ then the function $u(x)$ can be chosen so that $b_1 = 0$. The arc (6:9) then defines a conjugate interval on $x'x''$.

7. *A sub-problem of* P^0. Let $x = x^s$ $(s = 0,\cdots,r+1;$ $x^0 = x_1$, $x^{r+1} = x_2)$ be a set of points arranged in the order of their superscripts. In this section we shall consider the problem P^* determined by a functional of the form (1:1) and a set of arcs $\mathcal{E}^*$ in $\mathcal{D}$ which vanish at the points x^s. Clearly $\mathcal{E}^*$ is a subset of $\mathcal{E}^0$ and P^* is a sub-problem of P^0 where $\mathcal{E}^0$ and P^0 are defined as in Section 4. It is to be noted that the extremal arcs for P^* are broken extremal arcs for P^0 with corners at the interior points x^s $(s = 1,\cdots,r)$. We shall assume that the strengthened Clebsch condition holds on x_1x_2.

For convenience we shall define the index of a sub-interval x^sx^{s+1} $(s = 0,\cdots,r)$ to be the number of arcs in a maximal set of arcs in $\mathcal{E}^*$ identically zero except on the interval x^sx^{s+1} $(s = 0,\cdots,r)$ every proper linear combination of which has $J(\eta) < 0$. The index of x^sx^{s+1} $(s = 0,\cdots,r)$ is then equal to the number of conjugate intervals of x^s interior to this interval.

THEOREM 7:1. *The index of* P^* *is equal to the sum of the indices of the sub-intervals* x^sx^{s+1} $(s = 0,\cdots,r)$.

For, let η be any arc in $\mathcal{E}^*$ having $J(\eta) < 0$. It is readily seen that η is equal to a sum of arcs in $\mathcal{E}^*$ each being identically zero except on an interval $x^s x^{s+1}$ and each having $J < 0$.

Let P be the problem defined in Section 4. We then have

THEOREM 7:2. *The index* q^* *of* P *with respect to* P^* *is equal to the number of broken extremal arcs* η_α $(\alpha = 1,\cdots,q^*)$ *with corners at the interior points* x^s $(s = 1,\cdots,r)$ *in a maximal set of arcs in* $\mathcal{E}$ *every linear combination* η *of which has* $J(\eta) \leqq 0$ *but no proper linear combination of which is a transversal arc for* P.

This is the same as Theorem 5:1 stated for a different type of sub-problem and it is easily verified that the theorem holds in this case.

COROLLARY. *If there exist no extremal arcs in* $\mathcal{E}^*$, *then the index* q^* *of* P *with respect to* P^* *is equal to the number of broken extremal arcs* η_α $(\alpha = 1,\cdots,q^*)$ *with corners at* x^s $(s = 1,\cdots,r)$ *in a maximal set of arcs in* $\mathcal{E}$ *every proper linear combination* η *of which has* $J(\eta) < 0$.

THEOREM 7:3. *If the points* x^s $(s = 0,\cdots,r+1)$ *are such that there exist no conjugate points of* x^s *on* $x^s x^{s+1}$ $(s = 0,\cdots,r)$, *then the index of* P *is equal to the number* k *of broken extremal arcs with corners at the interior points* x^s $(s = 1,\cdots,r)$ *in a maximal set of arcs in* $\mathcal{E}$ *every proper linear combination* η *of which has* $J(\eta) < 0$.

For, the index of each sub-interval $x^s x^{s+1}$ $(s = 0,\cdots,r)$ is equal to zero. Hence by Theorem 7:1 the index of P^* is equal to zero and, therefore, the index of P is the same as the index of P with respect to P^*. This number k is equivalent to the negative index of the index form of Morse (VIII).

LEMMA 7:1. *A finite number* r *of points* x^s $(s = 1,\cdots,r)$ *can always be chosen on* x_1x_2 *such that there exist no pairs of* conjugate points on $x^s x^{s+1}$ $(s = 0,\cdots,r)$.

For, suppose that the lemma is false. Choose arbitrarily a finite number of points on x_1x_2. The lemma is then false on at least one of the sub-intervals thus formed. On this interval make a further subdivision. It follows that the lemma is false on at least one of these sub-intervals. In this manner continue to make subdivisions of the interval x_1x_2. The points of subdivision will have a limit point x' and it follows that in every neighborhood of x' there exist pairs of conjugate points. But this contradicts the known fact that for any point x'' on x_1x_2 there exists an interval $x'' - \delta \leqq x \leqq x'' + \delta$ on which there are no pairs of conjugate points (II, 739). Hence the lemma is true.

THEOREM 7:4. If the strengthened Clebsch condition holds on x_1x_2 then the index k of P is finite.

For, by Lemma 7:1 we can choose the points x^s on x_1x_2 so that there exist no conjugate points of x^s on $x^s x^{s+1}$ $(s = 1,\cdots,r)$. Let r be the number of sub-intervals $x^s x^{s+1}$. Then the number k of Theorem 7:3 satisfies the inequality $k \leqq nr$ and hence the index of P is finite.

Let x^s $(s = 0,\cdots,r+1)$ be a set of points as described in the first paragraph of this section. Let t be the number of broken extremal arcs ζ_α $(\alpha = 1,\cdots,t)$ in $\mathcal{E}$ with corners at the interior points x^s $(s = 1,\cdots,r)$ every proper linear combination ζ of which has $J(\zeta) < 0$ and let v be the number of linearly independent arcs η_β $(\beta = 1,\cdots,v)$ vanishing at x^s $(s = 0,\cdots,r+1)$ which are transversal arcs for P, where P is defined as in Section 4. We then have the following

THEOREM 7:5. The index k of P is expressible in the form

$$k = p_0 + \cdots + p_r + t - v,$$

where p_s *is the number of primary conjugate points of* x^s *on* $x^s x^{s+1}$ $(s = 0, \cdots, r)$ *and* t *and* v *are defined as above.*

In order to prove this let η_γ^s $(\gamma = 1, \cdots, p_s; s = 0, \cdots, r)$ be a maximal set of broken extremal arcs determining the primary conjugate points of x^s $(s = 0, \cdots, r)$ on $x^s x^{s+1}$. These may be chosen so as to be identically zero except on $x^s x^{s+1}$. By the types of proofs used before it is readily verified that every arc η in $\mathcal{E}$ satisfying the conditions

$$(7{:}1) \qquad J(\eta_\gamma^s, \eta) = 0, \qquad J(\zeta_\alpha, \eta) = 0$$

has $J(\eta) \geqq 0$. Let $\bar\eta_\delta$ $(\delta = p_0 + \cdots + p_r - v)$ be a maximal set of the arcs η_γ^s no proper linear combination of which is a transversal arc for P. The conditions $J(\bar\eta_\delta, \eta) = 0$, $J(\zeta_\alpha, \eta) = 0$ then form a minimal set, for, every arc η in $\mathcal{E}$ satisfying these conditions also satisfies the conditions (7:1) and hence has $J(\eta) \geqq 0$. Moreover every linear combination ζ of the arcs $\bar\eta_\delta$, ζ_α has $J(\zeta) \leqq 0$ but no proper linear combination is a transversal arc for P. Hence the statement of the theorem is true.

Results analagous to those of this section may be verified for sub-problems P^* of P obtained by imposing the conditions $A_{1\alpha}^s \eta_1(x^s) = 0$ where the $A_{1\alpha}^s$ are constants and the points x^s are finite in number and lie interior to the interval x_1x_2.

THEOREM 7:6. *If the index* k *of* $J(\eta)$ *on* $\mathcal{E}$ *is finite then the Clebsch condition holds on* x_1x_2.

For, since k is finite and the index q^0 of P with respect to P^0 is finite, it follows that the index k^0 of $J(\eta)$ on $\mathcal{E}^0$ is also finite. Define the index of a sub-interval $x'x''$ of x_1x_2 to be the number of arcs in a maximal set of arcs in $\mathcal{E}^0$ identically zero except on $x'x''$ every proper linear combination η of which has $J(\eta) < 0$. The index of $x'x''$ is clearly less than or equal to the index of $J(\eta)$ on $\mathcal{E}^{0}$. It follows that the interval x_1x_2 can be subdivided into a finite number of intervals of length less than δ such that on all but a finite number of these the index is equal to zero. On those intervals for which the index is zero the arc $(\eta) \equiv (0)$ is a minimizing arc and hence the Clebsch condition holds. Since δ may be taken arbitrarily small it follows by continuity considerations that the Clebsch condition holds on the entire interval x_1x_2.

8. *Oscillation and comparison theorems.* We shall assume that the strengthened Clebsch condition holds on the interval x_1x_2. Let P and P^0 be problems as defined in Section 4. As a consequence of Corollary 2 of Theorem 6:3 we have the following general oscillation theorem.

THEOREM 8:1. *Let* k *be the index of* P *and* q^0 *the index of* P *with respect to* P^0. *There exist exactly* $k - q^0$ *conjugate intervals of* x_1 *or of* x_2 *interior to* x_1x_2.

As a consequence of the corollary to Lemma 5:2 we have

COROLLARY. *The number* k^0 *of conjugate intervals of* x_1 *interior to* x_1x_2 *satisfies the inequality*

$$k - 2n + p \leqq k^0 \leqq k.$$

Let P^* be a problem determined by a functional J^* and a set of end conditions $\Psi_\tau^* = 0$ $(\tau = 1,\cdots,p^*)$. We then have the following comparison theorems.

THEOREM 8:2. *Let* k, k^* *denote the indices and* s, s^* *the number of conjugate intervals for* x_1 *interior to* x_1x_2 *for* P *and* P^*, *respectively. Let* q, q^* *denote respectively the indices of* P *and* P^* *with respect to* P^0. *We then have the relations*

$$k - k^* = q - q^* + s - s^*,$$
$$|k - k^*| \leqq |s - s^*| + \max[2n - p,\ 2n - p^*].$$

The first result follows from Corollary 2 of Theorem 6:3. The second statement then follows as a consequence of the corollary of Lemma 5:2.

COROLLARY. *If* $\omega(\zeta, \zeta') = \omega^*(\zeta, \zeta')$, *then*

$$k - k^* = q - q^*,$$
$$|k - k^*| \leqq \max[2n - p,\ 2n - p^*] \leqq 2n.$$

These statements follow from Theorem 8:2 since $s - s^* = 0$.

Now suppose that P_1, P_1^* are sub-problems of P, P^* respectively and of the type studied in Section 4. Let k_1, k_1^* denote the indices and p_1, p_1^* the number of end conditions for P_1, P_1^*, respectively. We then have

THEOREM 8:3. *Let* q_1, q_1^* *denote the indices of* P, P^* *with respect to* P_1, P_1^*, *then*

$$k - k^* = q_1 - q_1^* + k_1 - k_1^*,$$
$$|k - k^*| \leqq |k_1 - k_1^*| + \max[p_1 - p,\ p_1^* - p^*].$$

If the sub-problems P_1, P_1^* *are identical, then*

$$k - k^* = q_1 - q_1^*,$$
$$|k - k^*| \leqq \max[p_1 - p,\ p_1^* - p^*] \leqq 2n.$$

If the problems P, P^* *are identical, then*

$$k_1 - k_1^* = q_1^* - q_1,$$
$$|k_1 - k_1^*| \leqq \max[p_1 - p,\ p_1^* - p] \leqq 2n.$$

Now consider two problems P, P^* having the same set of arcs $\mathcal{E}$ and let

$$J'(\eta) = J(\eta) - J^*(\eta).$$

The problem determined by $J'(\eta)$ and the set $\mathcal{E}$ shall be denoted by $P' = P - P^*$.

THEOREM 8:4. If k' is the index of P' then

$$k \leqq k^* + k'.$$

In order to prove this we shall suppose that k' is finite, for, the theorem is surely true if k' is infinite. Let the conditions

$$J'(\xi_\alpha, \eta) = 0 \qquad (\alpha = 1,\cdots,k') \tag{8:1}$$

form a proper minimal set for the problem P'. It follows that for every arc η in $\mathcal{E}$ satisfying these conditions $J(\eta) \geqq J^*(\eta)$. Let η_γ $(\gamma = 1,\cdots,h)$ be a maximal set of arcs in $\mathcal{E}$ satisfying these conditions every linear combination ζ of which has $J(\zeta) < 0$. Every arc η in $\mathcal{E}$ is expressible as a linear combination of the arcs ξ_α and an arc satisfying conditions (8:1). Clearly then $k \leqq k' + h$. Since the arcs η_γ satisfy (8:1), every linear combination $\eta = a_\gamma \eta_\gamma$ has $J^*(\eta) < 0$. Hence $h \leqq k^*$ and $k \leqq k^* + k'$.

THEOREM 8:5. Let k, k^* be the indices of P, P^* respectively and let d, d^* denote the maximum number of linearly independent transversal arcs in $\mathcal{E}$ for P, P^* respectively. If $J(\eta) \geqq J^*(\eta)$ for every arc $(\eta) \neq (0)$ in $\mathcal{E}$, then

$$k^* \geqq k, \qquad k^* + d^* \geqq k + d.$$

Moreover, if $J(\eta) > J^*(\eta)$ for all such arcs, then

$$k^* \geqq k + d.$$

In order to prove this let

$$J(\xi_\alpha, \eta) = 0 \qquad (\alpha = 1,\cdots,k)$$

be a minimal set for P and let ξ_γ ($\gamma = k+1,\cdots,k+d$) be a maximal set of linearly independent transversal arcs for P in $\mathcal{E}$. We then have

$$J(\xi_\alpha a_\alpha) \leqq 0 \qquad (\alpha = 1,\cdots,k+d),$$

the inequality holding only in case the first k a's are zero. It follows that

$$J^*(\xi_\alpha a_\alpha) \leqq 0 \qquad (\alpha = 1,\cdots,k+d),$$

the strict inequality holding in case the first k a's are not all zero. Hence $k^* \geqq k$. We now consider a maximal set $\bar{\xi}_\rho$ of the ξ_α ($\alpha = 1,\cdots,k+d$) no proper linear combination of which is a transversal arc for J^*. The number of arcs in the set $\bar{\xi}_\rho$ is at least $k + d - d^*$. Since these arcs form a subset of a set of functions defining a minimal set for P^* it follows that $k^* \geqq k + d - d^*$. Hence the second inequality is proved. If $J(\eta) > J^*(\eta)$ for every arc $(\eta) \not\equiv (0)$ in $\mathcal{E}$ then $J^*(\xi_\alpha a_\alpha) < 0$ ($\alpha = 1,\cdots,k+d$) for every set of constants $(a) \neq (0)$. Hence $k^* \geqq k+d$.

Now let P be the problem determined by the functional

$$J(\eta) = 2q_1(\eta_1) - 2q_2(\eta_2) + \int_{x_1}^{x_2} 2\omega(\eta, \eta')dx$$

and a set of arcs $\mathcal{E}$, the maximal set of arcs in $\mathcal{B}$ satisfying the end conditions

$$\begin{aligned} \Psi_{\mu 1}(\eta_1) &= a_{\mu i}\eta_i(x_1) = 0 \qquad (\mu = 1,\cdots,p_1 \leqq n), \\ \Psi_{\tau 2}(\eta_2) &= a_{\tau i}\eta_i(x_2) = 0 \qquad (\tau = 1,\cdots,p_2 \leqq n), \end{aligned} \tag{8:2}$$

where $2q_1$ and $2q_2$ are symmetric quadratic forms in η_{i1}, η_{i2} respectively. The transversality conditions are then expressible in the form

$$\begin{aligned} T_{i1}(\eta_1, \lambda, \chi_1) &= q_1\,\eta_{i1} + \chi_{\mu 1} a_{\mu i} - \zeta_i(x_1) = 0, \\ T_{i2}(\eta_2, \lambda, \chi_2) &= q_2\,\eta_{i2} + \chi_{\tau 2} a_{\tau i} + \zeta_i(x_2) = 0. \end{aligned} \tag{8:3}$$

A sub-problem P_1 may be obtained by replacing the end conditions $\Psi_{\tau 2} = 0$ by $\eta_i(x_2) = 0$. Also by replacing the end conditions $\Psi_{\mu 1} = 0$ by $\eta_i(x_1) = 0$ a sub-problem P_2 may be obtained.

Let P^* be a sub-problem of the problem P defined in the preceding paragraph. Suppose P^* is determined by a set of arcs $\mathcal{E}^*$ consisting of those arcs in $\mathcal{E}$ vanishing at x'. Then P^* is a sub-problem of the type studied in Section 7 having variable end points instead of fixed end points. Let $q(x')$ denote the number of broken extremal arcs η_α ($\alpha = 1,\cdots,q(x')$) in a maximal set of arcs in $\mathcal{E}$ having a single corner at x' every linear combination η of which has $J(\eta) \leqq 0$ and satisfies the transversality conditions for P but no proper linear combination of which is a transversal arc for P. If t_1 is equal to the number of focal intervals of x_1 interior to x_1x' and t_2 is equal to the number of focal intervals of x_2 interior to $x'x_2$ then the index k of P is expressible in the form

$$k = t_1 + t_2 + q(x'). \tag{8:4}$$

Let η_{ik} be the extremal arcs of a conjugate system determined by the first p_1 of conditions (8:2) and (8:3) and let u_{ik} be the extremal arcs of a conjugate system determined by the last p_2 of conditions (8:2) and (8:3). Let ζ_ρ ($\rho = 1,\cdots,r_1$) be a maximal set of linearly independent extremal arcs of the system η_{ik} which vanish at x' and ζ_γ ($\gamma = r_1+1,\cdots,r_1+r_2=r$) a maximal set or linearly independent extremal arcs of the system u_{ik} which vanish at x' no linear combination of which belongs to η_{ik}. It follows that the matrix $\|\zeta_{i\delta}(x')\|$ ($\delta = 1,\cdots,r$) has maximum rank r, where the functions $\zeta_{i\delta}$ are those associated with the extremal arcs $\xi_{i\delta}$. Define $\bar{\xi}_\rho \equiv \xi_\rho$ on x_1x', $\bar{\xi}_\rho \equiv 0$ elsewhere; $\bar{\xi}_\gamma \equiv \xi_\gamma$ on $x'x_2$, $\bar{\xi}_\gamma \equiv 0$ elsewhere. The r arcs $\bar{\xi}_\delta$ thus formed are then broken extremal arcs in $\mathcal{E}^*$. Since $J(\bar{\xi}_\delta, \eta) = 0$ for all broken extremal arcs η in $\mathcal{E}^*$ with corners at x = x' it follows that

$$\zeta_{i\delta}(x')\,\eta_i(x') = 0$$

for all such arcs η. This system of equations has n - r linearly independent non-zero solutions. Since on the interval x_1x' the arcs η_α described above are linear combinations of the arcs η_{ik} and on $x'x_2$ they are linear combinations of the arcs u_{ik}, it now follows that $q(x') \leqq n - r + r - d = n - d$, where d is the maximum number of linearly independent transversal arcs for P no proper linear combination of which vanishes at x'. A stronger but more complicated inequality can be obtained. However, the one obtained here is sufficient for the arguments to be given in Section 10. It is clear that if x' is not a focal point of either system then d is equal to the maximum number of linearly independent transversal arcs for P.

THEOREM 8:6. *If t_1 and t_2 denote the number of focal intervals of x_1 and x_2, respectively, interior to the interval x_1x_2, then the following relation holds:*

$$t_1 - t_2 = q(x_1 + 0) - q(x_2 - 0).$$

Moreover $|t_1 - t_2| \leqq n$.

For, in the expression (8:4) we let x' approach x_1 and then let x' approach x_2 obtaining the relations $k = t_2 + q(x_1 + 0)$ and $k = t_1 + q(x_2 - 0)$, respectively. The first relation of the theorem then follows immediately. The last relation is true since both $q(x_1 + 0)$ and $q(x_2 - 0)$ are less than or equal to n.

Let η_{ik}, ζ_{ik} and u_{ik}, v_{ik} be a pair of conjugate systems. We then have the following

THEOREM 8:7. *If* t_1 *and* t_2 *denote respectively the number of focal intervals of the conjugate systems* η_{ik}, ζ_{ik}, *and* u_{ik}, v_{ik} *on a given interval* γ *then the following relations hold*

$$t_1 - t_2 = q(x_1 + 0) - q(x_2 - 0) \quad \underline{\text{on}} \quad (x_1 < x < x_2),$$
$$t_1 - t_2 = q(x_1 + 0) - q(x_2 + 0) \quad \underline{\text{on}} \quad (x_1 < x \leqq x_2),$$
$$t_1 - t_2 = q(x_1 - 0) - q(x_2 - 0) \quad \underline{\text{on}} \quad (x_1 \leqq x < x_2),$$
$$t_1 - t_2 = q(x_1 - 0) - q(x_2 + 0) \quad \underline{\text{on}} \quad (x_1 \leqq x \leqq x_2),$$

where $q(x')$ *is defined as before for the problem* P *determined by* η_{ik} *and* u_{ik}. *In each of these cases* $|t_1 - t_2| \leqq n$.

The first relation follows immediately from Theorem 8:6. In order to obtain the second relation we consider the problem P defined on the interval x_1x_3 where x_3 is to the right of x_2 but sufficiently near so that the number of focal intervals of η_{ik} and u_{ik} on x_1x_3 is the same as on x_1x_2. By the same type of argument used to obtain the first relation we can now obtain the second. The remaining relations can be verified similarly.

9. *The accessory boundary value problem.* Let $J(\eta)$ be defined as in Section 1 and let $\mathcal{E}$ be the class of arcs as defined in Section 2. We shall consider the functional

$$J(\eta, \sigma) = J(\eta) - \sigma\int_{x_1}^{x_2} \eta_i \eta_i dx$$

called the *characteristic functional* of $J(\eta)$. A value of σ for which there exists a non-zero transversal arc η_i for the problem determined by $J(\eta, \sigma)$ on $\mathcal{E}$ is called a *characteristic value*. The functions $\eta_i(x)$ defining this transversal arc are said to form a *characteristic arc*. Associated with a given characteristic arc η are a set of multipliers $\lambda_\mu(x)$, χ_ν such that the Euler equations

$$L_i(\eta) - \sigma\eta_i = 0$$

and the transversality conditions

$$T_{i1}(\eta, \chi, \lambda) = 0, \qquad T_{i2}(\eta, \chi, \lambda) = 0$$

for the problem are satisfied. The functions L_i, T_{i1}, T_{i2} are defined as in equations (2:1) and (2:2).

In this section let $k(\sigma)$ denote the index of $J(\eta, \sigma)$ on $\mathcal{E}$ and let $d(\sigma)$ denote the number of linearly independent characteristic arcs associated with σ. The number $d(\sigma)$ will be called the order of σ as a characteristic value. It is understood that if σ is not a characteristic value then $d(\sigma) = 0$.

In order to establish a relationship between the index of $J(\eta)$ on $\mathcal{E}$ and the characteristic values of $J(\eta, \sigma)$ we prove the following lemmas. We shall assume that the strengthened Clebsch condition holds on x_1x_2.

LEMMA 9:1. *The index* $k(\sigma)$ *of* $J(\eta, \sigma)$ *on* $\mathcal{E}$ *is equal to zero for* $-\sigma$ *sufficiently large.*

In order to prove this we first note that by the theory of quadratic forms there exists a constant c such that $2q(\eta)$ in (1:1) satisfies the relation $2q(\eta) \geqq c(\eta_{i1}\eta_{i1} + \eta_{i2}\eta_{i2})$. Let $h(x)$ be any function of x of class C' on x_1x_2 such that $h(x_1) = -1$, $h(x_2) = +1$. Then

$$J(\eta, \sigma) \geqq \int_{x_1}^{x_2} [c \frac{d}{dx}(\eta_i \eta_i h) + 2\omega(x, \eta, \eta') - \sigma \eta_i \eta_i] dx$$

(9:1)

$$= \int_{x_1}^{x_2} [P^*_{ik} \eta_i \eta_k + 2Q^*_{ik} \eta_i \eta'_k + R_{ik} \eta'_i \eta'_k - \sigma \eta_i \eta_i] dx,$$

where P^*_{ik}, Q^*_{ik} are functions having the properties of P_{ik}, Q_{ik} (VII, 50-51). Now consider

$$Q = P^*_{ik} \eta_i \eta_k \tau^2 + 2Q^*_{ik} \eta_i \zeta_k \tau + R_{ik} \zeta_i \zeta_k + \eta_i \eta_i,$$

where η_i, ζ_i satisfy

(9:2) $$S_{\nu i} \eta_i \tau + T_{\nu i} \zeta_i = 0.$$

When τ is zero then ζ_i satisfies $T_{\nu i} \zeta_i = 0$ and $Q > 0$ since the strengthened Clebsch condition holds. By continuity considerations it follows that for functions η_i, ζ_i satisfying (9:2) we have $Q > 0$ for τ near zero. Now let $\zeta = \eta' \tau$ where τ is near zero, then

$$Q = P^*_{ik} \eta_i \eta_k \tau^2 + 2Q^*_{ik} \eta_i \eta'_k \tau^2 + R_{ik} \eta'_i \eta'_k \tau^2 + \eta_i \eta_i > 0.$$

Hence $Q \tau^{-2} > 0$. Setting $\sigma = \tau^{-2}$ in (9:1) it follows that the integrand becomes $Q \tau^{-2}$, and hence $J(\eta, \sigma) > 0$ for $-\sigma$ sufficiently large.

LEMMA 9:2. _If σ is sufficiently near a fixed value σ_0 then_

$$k(\sigma) = k(\sigma_0) \quad (\sigma < \sigma_0), \qquad k(\sigma) = k(\sigma_0) + d(\sigma_0) \quad (\sigma > \sigma_0).$$

Moreover, on an interval $\sigma_1 \leqq \sigma < \sigma_2$ there are exactly $k(\sigma_2) - k(\sigma_1)$ characteristic values where a characteristic value is counted a number of times equal to its order.

For, if $\sigma < \sigma_0$ then $J(\eta, \sigma) > J(\eta, \sigma_0)$. Hence by Theorem 8:5 we have $k(\sigma_0) \geqq k(\sigma) + d(\sigma)$. By virtue of this inequality there can be at most a finite number of characteristic values on an interval $\sigma_1 \leqq \sigma \leqq \sigma_2$. Otherwise $k(\sigma)$ would become infinite as σ varies from σ_1 to σ_2 and this is impossible. Hence the characteristic values are isolated. Now let $\xi_\alpha(x)$ $(\alpha = 1,\cdots, k(\sigma_0))$ be a set of arcs in $\mathcal{E}$ and having $J[\xi_\alpha, \xi_\beta, \sigma]a_\alpha a_\beta < 0$ $(\alpha, \beta = 1,\cdots,k(\sigma_0))$ for $\sigma = \sigma_0$, where $J(\xi, \eta, \sigma)$ is a functional of the type (3:1) formed for $J(\eta, \sigma)$. Since this relation holds for σ near σ_0 it follows that $k(\sigma) \geqq k(\sigma_0)$. Hence $k(\sigma) = k(\sigma_0)$. Also for σ sufficiently near σ_0 we have $d(\sigma) = 0$ since σ is not a characteristic value.

If $\sigma > \sigma_0$ then by the preceding argument $k(\sigma) \geqq k(\sigma_0) + d(\sigma_0)$. In order to show that $k(\sigma) \leqq k(\sigma_0) + d(\sigma_0$ choose points x^s $(s = 0,\cdots,h+1)$ with $x^0 = x_1$, $x^{h+1} = x_2$ arranged in order of their superscripts on x_1x_2 such that there exist no conjugate intervals of x^s on x^sx^{s+1} $(s = 0,\cdots,h)$. Let $\eta_\gamma(x, \sigma)$ $(\gamma = 1,\cdots,r)$ be a maximal set of r linearly independent broken extremal arcs in $\mathcal{E}$ continuous in σ and with corners at the points x^s $(s = 1,\cdots,h)$. The index $k(\sigma)$ and the order $d(\sigma)$ of σ as a characteristic value are equal respectively to the negative index and nullity of the quadratic form

$$J[\eta_\gamma(\sigma), \eta_\delta(\sigma), \sigma]b_\gamma b_\delta \qquad (\gamma, \delta = 1,\cdots,r).$$

As seen above $d(\sigma) = 0$ for σ sufficiently near σ_0. Hence as σ increases or decreases from σ_0 the index $k(\sigma)$ increases by at most $d(\sigma_0)$, that is,

$$k(\sigma) \leqq k(\sigma_0) + d(\sigma_0).$$

It follows that $k(\sigma) = k(\sigma_0) + d(\sigma_0)$ if $\sigma > \sigma_0$. Let the characteristic values on $\sigma_1\sigma_2$ be $\sigma^1 = \sigma_1$, $\sigma^2, \cdots, \sigma^q = \sigma_2$

and suppose $\sigma^1 < \sigma^2 < \cdots < \sigma^q$. Since $k(\sigma^s) = k(\sigma^{s-1}) + d(\sigma^{s-1})$ $(s = 2,\cdots,q)$, it follows that

$$k(\sigma_2) - k(\sigma_1) = d(\sigma^1) + \cdots + d(\sigma^{q-1})$$

and hence the last statement of the lemma is proved. It is clear that if $k(\sigma_1) = 0$ then $k(\sigma_2)$ is equal to the number of characteristic values less than σ_2. This follows from the above formula since for $\sigma < \sigma_1$, $k(\sigma) = k(\sigma_1) = 0$ and $d(\sigma) = 0$.

THEOREM 9:1. *If the strengthened Clebsch condition holds on* x_1x_2 *then the index* k *of* $J(\eta)$ *on* $\mathcal{C}$ *is equal to the number of negative characteristic values of its characteristic functional* $J(\eta, \sigma)$.

For, by Lemma 9:1 there exists a value of σ such that $k(\sigma) = 0$. Hence, as was noted in the proof of Lemma 9:2 there exist $k(0)$ characteristic values which are less than $\sigma = 0$, that is, the index of $J(\eta)$ is equal to the number of negative characteristic values of $J(\eta, \sigma)$.

10. *A pseudo-periodic problem*. We shall suppose that there exists a constant τ and a set of n^2 constants A_{ij} with $\det A = |A_{ij}| \neq 0$ such that the equations

$$2\omega(x, \eta, \eta') = 2\omega(x + \tau, u, u'), \tag{10:1}$$

$$\Phi_\nu(x, \eta, \eta') = \Phi_\nu(x + \tau, u, u') \tag{10:2}$$

are identities in x, η_i, η'_i, where

$$u_i = A_{ij}\eta_j, \qquad u_i' = A_{ij}\eta'_j .$$

As a result of these identities the functions L_i in the differential equations,

$$L_i(x, \eta, \eta', \eta'', \lambda) = \Omega_{\eta_i} - \frac{d}{dx}\Omega_{\eta'_i} = 0,$$

have the property that

$$L_i(x, \eta, \eta', \eta'', \lambda) = A_{ji}L_i(x + \tau, u, u', u'', \lambda) \tag{10:3}$$

identically in $x, \eta_i, \eta'_i, \eta''_i$, where $u''_i = A_{ij}\eta''_j$.

An extremal arc satisfying the conditions

$$\eta_i(x + \tau) = A_{ij}\eta_j(x), \qquad \zeta_i(x) = A_{ji}\zeta_i(x + \tau) \tag{10:4}$$

will be called a *pseudo-periodic arc* of period τ. As a consequence of the preceding identities any extremal arc satisfying (10:4) for one value of x satisfies these relations for all values of x.

The identities (10:1), (10:2), and (10:3) are still valid if we replace τ by $\alpha\tau$ and A_{ij} by A^{α}_{ij}, where A^{α}_{ij} are the elements in the matrix A^{α}. We shall call the problem $P_{\alpha\tau}$ determined by the functional

$$J(\eta) = \int_0^{\alpha\tau} 2\,\omega(x, \eta, \eta')\,dx$$

and a maximal class $\mathcal{E}_{\alpha\tau}$ of admissible arcs satisfying the conditions

$$\Phi_\nu(x, \eta, \eta') = 0,$$
$$\eta_i(\alpha\tau) = A^{\alpha}_{ij}\eta_j(0)$$

a *pseudo-periodic problem*. If the matrix with elements A^{α}_{ij} is the identity matrix then $P_{\alpha\tau}$ is the usual periodic problem. The transversality conditions for $P_{\alpha\tau}$ are given by

$$\zeta_i(0) = A^{\alpha}_{ji}\zeta_i(\alpha\tau).$$

It is seen that these conditions follow directly from (10:1).

THEOREM 10:1. *The number of conjugate intervals of* $(h-1)\tau$ *on* $(h-1)\tau < x < h\tau$ *is the same for all integral values of* h.

This follows as a consequence of the identities (10:3) and the fact that the conjugate intervals are unchanged by the transformation $u_i = A^{\alpha}_{ij}\eta_j$.

THEOREM 10:2. *The number of conjugate intervals on* $0 < x < p\tau$ *is equal to* p *times the number of conjugate intervals on* $0 < x < \tau$ *plus the number of broken extremal arcs in a maximal set of arcs vanishing at* 0 *and* $p\tau$, *having corners at the points* $h\tau$ $(h = 1,\cdots,p-1)$ *every linear combination of which has* $J \leqq 0$ *and such that no proper linear combination is an extremal arc.*

To prove this we consider the problem P^* determined by a functional of the form (10:1) defined on the interval $0 \leqq x \leqq p\tau$ and a set $\mathcal{E}^*$ of admissible arcs vanishing at $h\tau$ $(h = 0,\cdots,p)$ and satisfying $\Phi_\nu = 0$. As a consequence of Theorem 7:1 the index of P^* is equal to p times the number of conjugate intervals on $0 < x < \tau$. Since P^* is a sub-problem of a problem determined by a set of admissible arcs satisfying $\Phi_\nu = 0$ and vanishing at 0 and $p\tau$, the conclusion of the theorem follows as a consequence of Theorem 7:2.

THEOREM 10:3. *The index* k *of* $P_{\alpha\tau}$ *is given by the formula*

$$k = r_0 + r_1 + \cdots + r_{\alpha-1} + q,$$

where r_h $(h = 0,\cdots,\alpha-1)$ *is equal to the number of conjugate intervals of* $x = h\tau$ *on* $h\tau < x < (h+1)\tau$ *and* q *is equal to the number of broken extremal arcs with corners at* $x = (h+1)\tau$ $(h = 0,\cdots,\alpha-2)$, *in a maximal set of arcs in* $\mathcal{E}_{\alpha\tau}$ *every linear combination* η *of which has* $J(\eta) \leqq 0$, *no proper linear combination* *of which is a pseudo-periodic arc for* $P_{\alpha\tau}$.

LEMMA 10:1. *If* η_{ik}, ζ_{ik}, *is a conjugate system of extremals and*

$$u_{ik}(x + \alpha\tau) = A^{\alpha}_{ij}\,\eta_{jk}(x), \qquad \zeta_{ik}(x) = A^{\alpha}_{ji}\,v_{jk}(x + \alpha\tau),$$

then the functions u_{ik}, v_{ik} *define a conjugate system. Moreover, the number of focal intervals of* u_{ik} *on the interval* $x_1 + \alpha\tau < x \leqq x_2 + \alpha\tau$ *is equal to the number of focal intervals of* η_{ik} *on* $x_1 < x \leqq x_2$.

The proof of the first part of the lemma as given by Birkhoff and Hestenes (I, 241) is applicable in this case and is as follows. By virtue of (10:3), with τ replaced by $\alpha\tau$, the arcs u_{ik} are extremal arcs. Also

$$\eta_{ij}(x)\,\zeta_{ik}(x) = u_{ij}(x + \alpha\tau)\,v_{ik}(x + \alpha\tau),$$

and hence

$$0 = \eta_{ij}\zeta_{ik} - \eta_{ik}\zeta_{ij} = u_{ij}v_{ik} - u_{ik}v_{ij}.$$

It follows that the extremals u_{ik}, v_{ik} form a conjugate system. The last statement of the lemma is clearly true.

Let $\eta_{ik}(x)$ be the extremal arcs of a conjugate system η_{ik}, ζ_{ik} with $t_h(x_1)$ secondary focal points on the interval $x_1 < x \leqq x_1 + h\tau$. We then have

THEOREM 10:4. *The limit*

$$\mu = \lim_{h\to\infty} \frac{t_h(x_1)}{h}$$

exists, is independent of the conjugate system η_{ik}, ζ_{ik} *and the initial point* x_1. *Moreover, the inequalities*

$$(10:5) \qquad |h\mu - t_h(x_1)| \leqq n \qquad (h = 1, 2, \cdots)$$

are true for all values of x_1. *If* $\eta_{ik}(x_1) = 0$, *then*

$$t_h(x_1) \leqq h\mu \leqq t_h(x_1) + n \qquad (h = 1,2,\cdots). \tag{10:6}$$

To prove this let $\alpha = hk$ and

$$u_{ik}(x + (\alpha - 1)\tau) = A_{ij}^{\alpha - 1}\,\eta_{jk}(x).$$

By Lemma 10:1 the number of focal intervals of the system u_{ik} on $x_1 + (\alpha - h)\tau < x \leqq x_1 + \alpha\tau$ is equal to $t_h(x_1)$. By Theorem 8:7 the number of focal intervals of η_{ik} on this interval differs from $t_h(x_1)$ by at most n. Since this statement holds for all values of the integer k in $\alpha = hk$ we have the following relations:

$$\begin{aligned} &|t_\alpha(x_1) - t_{(k-1)h}(x_1) - t_h(x_1)| \leqq n, \\ &|t_{(k-1)h}(x_1) - t_{(k-2)h}(x_1) - t_h(x_1)| \leqq n, \\ &\quad \cdots \qquad \cdots \qquad \cdots \\ &|t_{2h}(x_1) - t_h(x_1) - t_h(x_1)| \leqq n. \end{aligned} \tag{10:7}$$

On adding these inequalities we have

$$|t_\alpha - kt_h(x_1)| \leqq (k - 1)n < kn.$$

Similarly

$$|t_\alpha - ht_k(x_1)| < hn.$$

On dividing by $\alpha = hk$ we have

$$\begin{aligned} 0 &\leqq \left| \frac{t_\alpha(x_1)}{\alpha} - \frac{t_h(x_1)}{h} \right| \leqq \frac{n}{h}, \\ 0 &\leqq \left| \frac{t_\alpha(x_1)}{\alpha} - \frac{t_k(x_1)}{k} \right| \leqq \frac{n}{k}, \end{aligned} \tag{10:8}$$

Hence

$$\left| \frac{t_h(x_1)}{h} - \frac{t_k(x_1)}{k} \right| \leqq \frac{n}{h} + \frac{n}{k}.$$

These relations hold for all values of x_1. By taking h and k sufficiently large the sum $\frac{n}{h} + \frac{n}{k}$ can be made arbitrarily small. Hence $\mu = \lim_{h\to\infty} \frac{t_h(x_1)}{h}$ exists and is the same for all values of x_1. Moreover if μ_1, μ_2 denote the limits of μ and $t_{1h}(x_1)$, $t_{2h}(x_1)$ denote the numbers $t_h(x_1)$ for two different conjugate systems then we have

$$|\mu_1 - \mu_2| = \lim_{h\to\infty} \frac{|t_{1h}(x_1) - t_{2h}(x_1)|}{h} \leqq \lim_{h\to\infty} \frac{n}{h} = 0.$$

It follows that the constant μ is the same for all conjugate systems. This constant will be called the frequency number. If we let k become infinite in the first of the relations (10:8) and note that $\lim_{k\to\infty} \frac{t_\alpha(x_1)}{\alpha} = \mu$ we get the inequality (10:5). If $\eta_{1k}(x_1) = 0$ then the number equivalent to q_2 in Lemma 6:5 is equal to zero. It follows that the expressions (10:7) and hence (10:8) hold when the absolute value signs are removed. By letting k become infinite we obtain the relations (10:6).

COROLLARY. If there exists a conjugate system η_{ik}, ζ_{ik} with η_{ik}, ζ_{ik} periodic, then the frequency number μ is rational.

For, if h is suitably chosen then

$$\eta_{ik}(x + (\alpha - 1)\tau) = A_{ij}^{\alpha - 1} \eta_{jk}(x),$$

where $\alpha = hk$ and $k = 1, 2, \cdots$. It follows that the left members of the inequalities (10:7) are equal to zero and hence

$$h\mu - t_h(x_1) = 0.$$

The conclusion of the corollary is immediate.

Finally let us consider the periodic problem with period τ where the matrix A is the identity matrix and let P be the problem defined on the interval $0 \leqq x \leqq r\tau$ with end conditions $\eta_i(0) = \eta_i(r\tau) = 0$. Let P^* be the sub-problem of P with arcs vanishing at $x = \tau$. This is a problem of the type discussed in Section 7 and the index of P with respect to P^* is the number of broken extremal arcs in a maximal set with corners at τ, vanishing at $x = 0$ and $x = r\tau$, every linear combination of which having $J(\eta) \leqq 0$, but no proper linear combination being a periodic extremal arc. This number is the same as the rth index of repletion as defined by Morse and Pitcher (X). They consider only the case when the points $x = \tau$ and $x = r\tau$ are not conjugate to each other nor to $x = 0$.

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MINIMA OF DOUBLE INTEGRALS WITH RESPECT TO UNILATERAL VARIATIONS AND APPLICATIONS TO SUBHARMONIC FUNCTIONS

BY

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MINIMA OF DOUBLE INTEGRALS WITH RESPECT TO UNILATERAL VARIATIONS, AND APPLICATIONS TO SUBHARMONIC FUNCTIONS

1. Introduction. A function $z(x,y)$ having continuous second order partial derivatives in an open region R of the xy-plane is harmonic in that region if it satisfies the partial differential equation of Laplace,

$$\Delta z \equiv \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

in R. This will be recognized as the Euler equation for the calculus of variations problem of minimizing the Dirichlet integral

$$I[z;R] \equiv \iint_R (z_x{}^2 + z_y{}^2)dxdy$$

in a class of "admissible" functions assuming specified values on the frontier R' of R. An analogous situation arises in the consideration of subharmonic functions. A function $u(x,y)$ having continuous second order partial derivatives in R is subharmonic there if and only if $\Delta u \geqq 0$ in this region [11, pp. 12, 13]*. If, in addition to satisfying these conditions, u is continuous in the closure $\bar{R}$ of R, then it is readily shown that u renders the Dirichlet integral a minimum in a class of "admissible" functions coinciding with u on R' and lying below u in R. The question arises as to whether this property is retained by subharmonic functions not having the differentiability properties indicated above for u. Another question is whether this property is characteristic for subharmonic functions. These questions suggest, and will be approached by way of, the consideration of a more general problem of the calculus of variations. This more general problem is that of investigating necessary and sufficient conditions for a given surface to furnish a minimum to a double integral in a class of surfaces lying below it interior to its region of definition, but having the same values on the frontier. It is the aim of the present paper to develop such conditions for this more general problem, and to indicate how some of the defining properties of subharmonic functions may be regarded as special cases. It will be found that many of the conditions derived • closely parallel the usual conditions of the calculus of variations for minima with respect to variations which are unrestricted in sign.

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*Numbers in brackets refer to the bibliography at the end of this paper.

Section 2 of this paper is prefatory. In Section 3 are reviewed some properties of integral mean functions which are extensively employed as approximating functions in succeeding developments. Most of these are known and may be found elsewhere in the literature. Section 4 presents the development of the above mentioned necessary and sufficient conditions for minima with respect to unilateral variations. The results obtained by specializing the integrand function in various ways are sufficiently engaging as to merit particular attention in Section 5. Applications to subharmonic functions and the Dirichlet integral are made in Section 6. Some characterizing properties for subharmonic functions are there obtained as special cases of the preceding theory. Some of these results are obtained in strengthened form on account of the special nature of subharmonic functions.

Section 7 involves the application of the theory of Sections 4 and 5 to the least area problem and the problem of Plateau.

2. Notation and preliminary remarks. Throughout this paper R will be used to denote a finite open region of the xy-plane. If E is any subset of R, $\bar{E}$ will denote the closure of E, while E' will represent the set of frontier points of E.

If (x_0,y_0) is a given point, $T(x_0,y_0;h)$ will signify the open square region with side 2h: $x_0 - h < x < x_0 + h$, $y_0 - h < y < y_0 + h$; similarly, $D(x_0,y_0;h)$ will denote the open circular region with center at (x_0,y_0) and radius h. We shall also write $Q(x_0,y_0;h)$ for $T'(x_0,y_0;h)$, and $C(x_0,y_0;h)$ in place of $D'(x_0,y_0;h)$.

Integration is to be in the sense of Lebesgue. The class of all measurable functions g(x,y) for which $|g|^n$ is summable in the region of definition of g will be represented, as is customary, by L^n. We shall also designate by L^∞ the class of measurable functions g(x,y) essentially bounded in their regions of definition; i.e., for each g(x,y) in L^∞ there exists a constant k such that $|g| < k$ almost everywhere in the region of definition of g. If g(x,y) is measurable in R, then we write for brevity,

$$M_n[g;R] \equiv [\iint_R |g|^n dxdy]^{1/n},$$

$$M[g;R] \equiv M_1[g;R],$$

$$M_\infty[g;R] \equiv \text{the essential upper bound of } g \text{ in } R.$$

Then, if $m^{-1} + n^{-1} = 1$ ($m = \infty$ when $n = 1$), and if f, g are in L^m, L^n, respectively, we have the Hölder inequality,

$$M[fg;R] \leqq M_m[f;R]M_n[g;R].$$

In the following pages the symbols m *and* n *will always be used, as above, to denote positive numbers having the property that* $m^{-1} + n^{-1} = 1$ ($m = \infty$ *when* $n = 1$). Then clearly, $m \geqq 1$, $n \geqq 1$.

Two measurable functions will be called *equivalent* in case they coincide except perhaps on a set of measure zero.

If V is an arbitrary condition or property, $E[V]$ signifies the set of all points (x,y) satisfying this condition or having this property. Thus, for example, $E[(x,y) \text{ in } R;\ y = y_0]$ denotes the set of points (x,y) in R for which $y = y_0$; i.e., the intersection of the line $y = y_0$ with R.

A function $f(x,y)$ is said to be of class $C^{(n)}$ in R if it is continuous in R together with its partial derivatives of the first n orders.

Consider the closed rectangle K: $a \leqq x \leqq b$, $c \leqq y \leqq d$. For a fixed y_0 in the interval (c,d) we shall denote by $V_a^b f(x,y_0)$ the total variation of $f(x,y_0)$ regarded as a function of x alone on the interval (a,b). A corresponding meaning is attached to the symbol $V_c^d f(x_0,y)$.

DEFINITION 2:1. *A continuous function* $f(x,y)$ *is absolutely continuous in the sense of Tonelli (abbreviated* A.C.T.) *on the rectangle* K *if*

(i) *it is an absolutely continuous function of* x *for almost every value of* y *on the interval* (c,d) *and an absolutely continuous function of* y *for almost every value of* x *on the interval* (a,b);

(ii) $V_c^d f(x_0,y)$ *and* $V_a^b f(x,y_0)$ *are summable functions of* x_0 *and* y_0 *on* $a \leqq x_0 \leqq b$ *and* $c \leqq y_0 \leqq d$, *respectively*.

It may be readily verified that $V_c^d f(x_0,y)$ and $V_a^b f(x,y_0)$ are lower semi-continuous functions of x_0 and y_0, respectively, on the respective intervals (a,b) and (c,d) [14, p. 169].

DEFINITION 2:2. *A continuous function* $f(x,y)$ *is* A.C.T. *in a region* R (*or* $\bar{R}$) *in case it is* A.C.T. *on every closed interior rectangle*.

DEFINITION 2:3. *A function* $f(x,y)$ *is Lipschitzian with constant* A *on a given point set in case for every pair of points* (x_1,y_1) *and* (x_2,y_2) *in the set*,

$$|f(x_1,y_1) - f(x_2,y_2)| \leqq A[(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2}.$$

Throughout the following presentation, we shall understand by the statement that a function $f(x,y)$ is absolutely continuous on an open linear set, or on the closure of an open linear set, that f is absolutely continuous on every closed interval contained in the linear set.

DEFINITION 2:4. *A function* $f(x,y)$ *defined in* R *will be called absolutely continuous by sections (abbreviated* a.c.s.) *in* R *if*

(i) $f(x,y)$ *is continuous in* R;

(ii) $f(x,y)$ *is an absolutely continuous function of* x *on the linear set* $E[(x,y)$ *in* $R;\ y = y_0]$ *for almost all* y_0;

(iii) $f(x,y)$ *is an absolutely continuous function of* y *on the linear set* $E[(x,y)$ *in* $R;\ x = x_0]$ *for almost all* x_0.

A function $f(x,y)$ satisfying in R the conditions of any one of the Definitions 2:2, 2:3, or 2:4 has first order partial derivatives which exist almost everywhere in R and are measurable in R. In particular, if $f(x,y)$ is Lipschitzian with constant A in R these partial derivatives are bounded by the Lipschitz constant A; moreover, $f(x,y)$ is A.C.T. in R.

3. *Integral means.* Let $f(x,y)$ be a function summable (whose Lebesgue integral exists and is finite) in an open region R of the xy-plane, and let S be an open region contained with its frontier S' in R. Finally, let h be a positive number so small that for $|a| \leq h$, $|b| \leq h$, and (x,y) in S, the point $(x + a, y + b)$ is in R. Under these conditions the quantity

$$f^{(h)}(x,y) = \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x+a,\ y+b)\,da\,db,$$

called *the integral mean of* f *on the square* $T(x,y;h)$, is well defined in S. A summary of properties of integral means to be employed in the developments of subsequent sections will now be presented. Most of these properties are well known and will be given without proof.

PROPERTY 3:1. *The function* $f^{(h)}(x,y)$ *is continuous in* S, *absolutely continuous in* x *for every* y, *and absolutely continuous in* y *for every* x.

PROPERTY 3:2. *If* $f(x,y)$ *is continuous in* R, *then* $f^{(h)}(x,y)$ *approaches* $f(x,y)$ *uniformly on* S *as* h *approaches zero. If* f *is only summable, then* $f^{(h)}$ *approaches* f *almost everywhere on* S.

PROPERTY 3:3. *Regarded as a function of four variables* (x,y,a,b), *the function* $f(x+a,\ y+b)$ *is summable in the region* $E[(x,y)$ *in* $S;\ |a| \leq h,\ |b| \leq h]$. *Hence, by Fubini's theorem,*

$$\iint_S f^{(h)}(x,y)\,dx\,dy = \frac{1}{4h^2} \iint_S \left\{ \int_{-h}^{h} \int_{-h}^{h} f(x+a,\ y+b)\,da\,db \right\} dx\,dy$$
$$= \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} \left\{ \iint_S f(x+a,\ y+b)\,dx\,dy \right\} da\,db.$$

PROPERTY 3:4. *Let* $k(x,y)$ *be a function belonging to the Lebesgue class* L^m $(1 \leq m \leq \infty)$ *in* R. *If* $f(x,y)$ *belongs to* L^n, *then* $k(x,y)f^{(h)}(x,y)$ *converges in the mean to* $k(x,y)f(x,y)$ *in* S *as* h *approaches zero.*

This statement is a consequence of Properties 3:1 and 3:2. To show this, let X be an arbitrary measurable set contained in R. Then, by Hölder's inequality,

$$\begin{aligned} &M[k(x,y)(f(x,y) - f(x+a,\ y+b));\ X] \\ &\qquad \leqq M[k(x,y)f(x,y);X] + M_m[k(x,y);X]M_n[f(x+a,\ y+b);X] \\ &\qquad \leqq M[kf;X] + M_m[k;X]M_n[f;X_0] \end{aligned}$$

where the set X_0 appearing in the final expression is merely a translation of X, and is contained in R. In view of the absolute continuity of the indefinite integral, the three integrals on the right in the above inequality tend to zero as meas. (X), the measure of X, approaches zero.

Now by Property 3:3,

$$M[k(f - f^{(h)});X] \leqq \frac{1}{4h^2}\int\!\!\int_X \left\{ \int_{-h}^{h}\int_{-h}^{h} |k(x,y)|\,|f(x,y) - f(x+a,y+b)|\,dadb \right\} dxd$$

$$(3{:}1) \qquad \leqq \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} M[k(x,y)(f(x,y) - f(x+a,y+b));X]\,dadb.$$

From the foregoing paragraph, it is seen that to every $\epsilon > 0$, there corresponds a δ_ϵ such that whenever $X \subset S$, and meas. $(X) < \delta_\epsilon$, it follows that

$$M[k(x,y)(f(x,y) - f(x+a,\ y+b));\ X] < \epsilon\ .$$

Consequently, for such sets X the right-hand member of (3:1) is also less than ϵ, and by this inequality the class of integrals $\left\{ \int\!\!\int |k(f - f^{(h)})|\,dxdy \right\}$ is seen to be uniformly absolutely continuous [5, vol. 2, p. 296] on S. Moreover, $\lim\limits_{h\to 0} |f - f^{(h)}| = 0$ almost everywhere in S by Property 3:2. Therefore [5, vol. 2, p. 297],

$$\lim_{h\to 0} M[k(f - f^{(h)});S] = M[\lim_{h\to 0} k(f - f^{(h)});S] = 0.$$

PROPERTY 3:5. *If* $f(x,y)$ *is summable in* R, *then* $f^{(h)}$ *converges in the mean to* $f(x,y)$ *in* S.

This follows at once from Property 3:4 by taking $k(x,y) \equiv 1$, $m = \infty$, $n = 1$.

PROPERTY 3:6. *If* $f(x,y)$ *is summable in* R, *then almost everywhere on any linear set* $E[(x,y)$ *in* $S;\ y =$ *const.*$]$ *the partial derivative* $\partial f^{(h)}/\partial x$ *exists and is given by*

$$\frac{\partial}{\partial x} f^{(h)} = \frac{1}{4h^2}\int_{-h}^{h}(f(x+h,y+b) - f(x-h,y+b))db = \frac{1}{4h^2}\int_{Q(x,y;h)} f\,dy \ ,$$

where the latter symbol signifies a line integral. Similarly, almost everywhere on $E[(x,y)$ *in* $S;\ x =$ *const.*$]$ *the partial derivative* $\partial f^{(h)}/\partial y$ *exists and is given by*

$$\frac{\partial}{\partial y} f^{(h)} = \frac{1}{4h^2}\int_{-h}^{h}(f(x+a,y+h) - f(x+a,y-h))da = -\frac{1}{4h^2}\int_{Q(x,y;h)} f\,dx.$$

In particular, these equalities hold almost everwhere in S.

PROPERTY 3:7. *Suppose that the function* $f(x,y)$ *is absolutely continuous in* x *for almost every* y *and that* $\partial f/\partial x$ *is summable on arbitrary closed subsets of* R. *Then*

$$\frac{\partial}{\partial x} f^{(h)} = \left(\frac{\partial f}{\partial x}\right)^{(h)} \tag{3:2}$$

in S. *In like manner, if* $f(x,y)$ *is absolutely continuous in* y *for almost every* x, *and if* $\partial f/\partial y$ *is summable on arbitrary closed subsets of* R, *then*

$$\frac{\partial}{\partial y} f^{(h)} = \left(\frac{\partial f}{\partial y}\right)^{(h)} \tag{3:3}$$

in S.

It follows immediately from this proposition that under the circumstances described $f^{(h)}$ has continuous first partial derivatives which possess all the properties of integral means presented in this section. To verify (3:2), it is first noted that Property 3:6 implies the validity of the equality almost everywhere on any linear set $E[(x,y) \text{ in } S;\ y = y_0]$, and hence, also, almost everywhere in S. Combining this fact with the absolute continuity of $f^{(h)}$ as a function of x for every y, and expressing $f^{(h)}$ as an indefinite integral, we have for any fixed $y = y_0$ and x_0 such that the interval (x_0,x) is in $E[(x,y) \text{ in } S;$ $y = y_0]$,

$$f^{(h)}(x,y_0) - f^{(h)}(x_0,y_0) = \int_{x_0}^{x} \frac{\partial}{\partial x} f^{(h)}(x,y_0)dx = \int_{x_0}^{x} \left(\frac{\partial f(x,y_0)}{\partial x}\right)^{(h)} dx.$$

The integrand of the last integral is continuous so that (3:2) holds everywhere on $E[(x,y) \text{ in } S;\ y = y_0]$. But y_0 is arbitrary, so that (3:2) is actually valid everywhere in S. Relation (3:3) may be established by a similar argument.

PROPERTY 3:8. *If* f(x,y) *is summable in* R, *then* $f^{(h)}(x,y)$ *is* A.C.T. *in its region of definition.*

Without loss of generality, we may restrict our attention to a rectangle K: $s \leqq x \leqq t,\ u \leqq y \leqq v$. Suppose that h_0 is a positive number chosen so that $(x+a, y+b)$ is in R whenever (x,y) is in K and $|a| \leqq h_0$, $|b| \leqq h_0$. For a fixed positive h such that $h \leqq h_0$ it follows by Property 3:6 that almost everywhere on a set $E[(x,y) \text{ in } K;\ y = y_0]$ we have

$$\left|\frac{\partial}{\partial x} f^{(h)}(x,y_0)\right| = \left|\frac{1}{4h^2} \int_{-h}^{h} (f(x+h,y_0+b) - f(x-h,\ y_0+b))db\right|$$

(3:4)

$$\leqq \frac{1}{4h^2} \int_{-h_0}^{h_0} |f(x+h,\ y_0+b) - f(x-h,\ y_0+b)|db.$$

For such a value h, and for any y_0 in (u,v), the point $(x+h, y_0+b)$ is in R whenever x is in (s,t) and $|b| \leqq h_0$. Consequently,

(3:5) $$J(h,y_0;h_0) \equiv \frac{1}{4h^2}\int_{s}^{t}\int_{-h_0}^{h_0} |f(x+h,y_0+b) - f(x-h,y_0+b)|dbdx$$

exists and is finite. Furthermore [5, vol. 1, p. 605],

$$(3{:}6)\quad V_s^t f^{(h)}(x,y_0) = \int_s^t \left|\frac{\partial}{\partial x} f^{(h)}(x,y_0)\right| dx, \quad (h \leqq h_0,\ y_0 \text{ in } (u,v)).$$

By (3:4) and (3:5) we thus obtain

$$V_s^t f^{(h)}(x,y_0) \leqq J(h,y_0;h_0), \qquad (h \leqq h_0,\ y_0 \text{ in } (u,v)).$$

Now $J(h, y_0; h_0)$ is an absolutely continuous function of y_0 on (u,v) for every $h \leqq h_0$. This is established by the same argument as that used in the proof of the absolute continuity of $f^{(h)}(x,y)$. Thus $J(h,y_0;h_0)$, considered as a function of y_0, is summable on (u,v). Therefore $V_s^t f^{(h)}(x,y_0)$ is a lower semi-continuous function of y_0 on (u,v), dominated by a summable function, and hence is itself summable. In a precisely similar fashion we may show the summability of $V_u^v f^{(h)}(x_0,y)$ with respect to x_0 on the interval (s,t). In accordance with Definitions 2:1 and 2:2 it now follows that $f^{(h)}(x,y)$ is A.C.T. in its region of definition. By (3:6), the summability of $V_u^v f^{(h)}(x_0,y)$ and of $V_s^t f^{(h)}(x,y_0)$ with respect to x_0 and y_0, respectively, implies the summability of $\partial f^{(h)}/\partial x$ and of $\partial f^{(h)}/\partial y$. Hence we have the following result.

PROPERTY 3:9. *If* $f(x,y)$ *is summable in* R, *then* $\partial f^{(h)}/\partial x$ *and* $\partial f^{(h)}/\partial y$ *are summable on arbitrary closed sets interior to their regions of definition.*

4. *Conditions for the minimum of a double integral with respect to unilateral variations.* The early portions of this section will be devoted to establishing some preliminary propositions which will be used in the proofs of later theorems.

LEMMA 4:1. *If* $g(x,y)$ *is summable in* R, *and*

$$\iint_R t(x,y)g(x,y)\,dx\,dy \geqq 0$$

for every function $t(x,y)$ *that is Lipschitzian and non-positive on* $\bar{R}$, *and which vanishes on* R', *then* $g(x,y) \leq 0$ *almost everywhere in* R.

This lemma is stated for Lipschitzian functions only for direct application to the proofs of Lemma 4:2 and Theorem 4:2. The following proof would apply almost verbatim if the functions $t(x,y)$ for which the inequality of the lemma is supposed to hold were restricted to have derivatives of arbitrarily high order.

Let $C(x_0,y_0;r)$ be a circle lying with its interior in R, and let d be a constant such that for $|a| \leq d$, $|b| \leq d$, the point $(x+a,\ y+b)$ is in R whenever (x,y) is in $D(x_0,y_0;r)$. Now select any Lipschitzian function $t(x,y)$ which vanishes identically on the complement of $D(x_0,y_0;r)$ and is non-positive in $D(x_0,y_0;r)$. For a fixed pair (a,b) satisfying $|a| \leq d$, $|b| \leq d$, define $t_1(x,y)$ by $t_1(x+a,\ y+b) = t(x,y)$. Then

$$(4:1)\quad \iint_{D(x_0,y_0;r)} t(x,y)g(x+a,\ y+b)dxdy = \iint_{D_1} t_1(x,y)g(x,y)dxdy \geq 0,$$

where $D_1 = D(x_0+a,\ y_0+b;r)$, in view of the hypothesis of the lemma. Integrating with respect to a and b, we thus obtain,

$$\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}\left\{\iint_{D(x_0,y_0;r)} t(x,y)g(x+a,y+b)dxdy\right\}dadb \geq 0\ ,\ (0 < h \leq d).$$

By Fubini's theorem, we may interchange the order of integration and conclude that

$$(4:2)\qquad \iint_{D(x_0,y_0;r)} t(x,y)g^{(h)}(x,y)dxdy \geq 0.$$

But $g^{(h)}(x,y)$ is continuous in $D(x_0,y_0;r)$ and (4:2) is valid for all $t(x,y)$ of the type described. Hence, it is easily seen that $g^{(h)}(x,y) \leq 0$ in $D(x_0,y_0;r)$ for every h satisfying the condition $h \leq d$. Therefore, $g(x,y) \leq 0$ almost everywhere in $D(x_0,y_0;r)$, by Property 3:2. And now, since R may be expressed as the sum, in the point set sense, of a denumerable number of circles, the desired conclusion is established.

Corresponding to any open set S comprised in R, $\mathcal{N}_S^m$ shall represent the class of functions $\eta(x,y)$ which are continuous in $\bar{R}$, a.c.s. in R and have the further properties:

(i) η_x, η_y belong to the class L^m;

(ii) η vanishes on the set $\bar{R} - S$.

For simplicity of notation we shall write $\mathcal{N}^m$ in place of $\mathcal{N}_R^m$. It follows at once from this definition that if $\eta(x,y)$ is in the class $\mathcal{N}_S^m$, then $\eta(x,y)$ is A.C.T. in $\bar{R}$. It is a further consequence of this definition that *if* $\eta(x,y)$ *is a non-positive function belonging to the class* $\mathcal{N}_S^m$, *then almost everywhere in* R - S *the derivatives* η_x *and* η_y *exist and are equal to zero*. Since these derivatives are clearly zero on $\bar{R}$ - S, it is only necessary to show that $\eta_x = \eta_y = 0$ almost everywhere on S'. Now, since η is A.C.T. in $\bar{R}$, the partial derivatives η_x, η_y exist almost everywhere in $\bar{R}$, and hence they exist almost everywhere on S'. But η attains a maximum at each point of S' since it is a non-positive function vanishing on S', so that almost everywhere on S' we have $\eta_x = \eta_y = 0$.

We next introduce a notion of "almost all squares" in a given region.

DEFINITION 4:1. *Let* $\mathcal{E}$ *be a class of squares contained with their interiors in an open region* R. *Corresponding to a given positive* h, *let* N_h *denote the set of all points* (x,y) *in* R *such that* $\bar{T}(x,y;h)$ *is contained in* R. *If for each* h *the set of points* (x,y) *of* R *for which* $\bar{T}(x,y;h)$ *is in* N_h *but not in* $\mathcal{E}$ *has measure zero, then* $\mathcal{E}$ *is said to constitute almost all squares of* R.

LEMMA 4:2. *Suppose that* $g_1(x,y)$ *is summable in* R, *that* $g_2(x,y)$, $g_3(x,y)$ *are functions belonging to the class* L^m *in* R, *and that for every non-positive* $\eta(x,y)$ *in the class* $\mathfrak{N}^n$ *the inequality*

$$(4{:}3)\qquad \iint_R (\eta g_1 + \eta_x g_2 + \eta_y g_3)dxdy \geqq 0$$

holds. Then, for almost all squares Q *of* R,

$$(4{:}4)\qquad \int_Q g_2dy - g_3dx \geqq \iint_T g_1dxdy.$$

Conversely, if g_1 *is summable in* R, *if* g_2, g_3 *belong to the class* L^m *in* R, *and if* (4:4) *holds for almost all squares of* R, *then* (4:3) *is true for every non-positive* η *in* $\mathfrak{N}^n$.

To verify the first part of this lemma, it is sufficient to confine our attention to Lipschitzian η's. Choose any open set S contained with its frontier S' in R. Let d be a positive number such that if $|a| \leqq d$, $|b| \leqq d$, then $(x+a,\ y+b)$ is in R whenever (x,y) is in S. For such pairs (a,b), and for Lipschitzian η's vanishing on S' it follows, by the same reasoning which led to (4:1) in the proof of Lemma 4:1, that

$$\iint_S [\eta(x,y)g_1(x+a,y+b) + \eta_x(x,y)g_2(x+a,y+b)$$
$$+ \eta_y(x,y)g_3(x+a,y+b)]dxdy \geqq 0.$$

Integrating this expression with respect to a and b between the limits (-h,h), where $0 < h \leqq d$, and interchanging the order of integration, we conclude that

$$(4{:}5)\qquad \iint_S (\eta g_1^{(h)} + \eta_x g_2^{(h)} + \eta_y g_3^{(h)})dxdy \geqq 0, \quad (0<h\leqq d).$$

Now, in view of Property 3:1, the functions $\eta g_2^{(h)}$ and $\eta g_3^{(h)}$ are a.c.s. and almost everywhere in S their derivatives are given by the usual formula for the derivative of a product. The functions $\eta_x g_2^{(h)}$ and $\eta_y g_3^{(h)}$ are clearly summable in S. Furthermore, since η is bounded in S, the summability of $\eta \partial g_2^{(h)}/\partial x$ and $\eta \partial g_3^{(h)}/\partial y$ is implied by that of $\partial g_2^{(h)}/\partial x$ and of $\partial g_3^{(h)}/\partial y$ [Property 3:9]. It therefore follows that the partial derivatives $\partial(\eta g_2^{(h)})/\partial x$ and $\partial(\eta g_3^{(h)})/\partial y$ are summable in S. Inequality (4:5) may accordingly be rewritten as follows:

$$\begin{aligned}&\iint_S \eta\,(g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)})dxdy\\ (4:6)\qquad &+ \iint_S [\frac{\partial}{\partial x}(\eta g_2^{(h)}) + \frac{\partial}{\partial y}(\eta g_3^{(h)})]dxdy \geqq 0, \quad (0<h\leqq d).\end{aligned}$$

Now $\eta g_2^{(h)}$ and $\eta g_3^{(h)}$ are a.c.s. in S, and vanish on S' since η vanishes there. Consequently, the second integral in (4:6) is zero, as can be easily seen if we first express it as an iterated integral, and then apply the fundamental theorem of the integral calculus for the Lebesgue integral. Hence

$$\iint_S \eta\,(g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)})dxdy \geqq 0, \qquad (0 < h \leqq d),$$

for every non-positive, Lipschitzian η vanishing on S'. And now, applying Lemma 4:1, it results that

$$(4:7)\qquad g_1^{(h)} - \frac{\partial}{\partial x} g_2^{(h)} - \frac{\partial}{\partial y} g_3^{(h)} \leqq 0, \qquad (0 < h \leqq d),$$

almost everywhere in S.

The set of points N_h appearing in Definition 4:1 is open. Therefore, for each h the inequality (4:7) holds by the foregoing proof almost everywhere in N_h. But by the definition of integral mean, and by Property 3:6, this is equivalent to the statement that for a fixed h, $0 < h \leqq d$ the inequality

$$(4:8)\qquad \int_{Q(x,y;h)} g_2dy - g_3dx \geqq \iint_{T(x,y;h)} g_1dxdy$$

holds almost everywhere in N_h. Then, by Definition 4:1, the class of squares for which (4:8) holds constitutes almost all squares of R.

The validity of the converse statement will be established in two steps. Under the assumption that (4:4) holds for almost all squares in R, we shall first prove that the inequality (4:3) is true when η is in the class $\mathfrak{N}_S^n$ and S is contained with its frontier in R. By Property 3:4,

$$\begin{aligned}\lim_{h\to 0}\iint_S (\eta g_1{}^{(h)} + \eta_x g_2{}^{(h)} + \eta_y g_3{}^{(h)})dxdy \\ = \iint_S (\eta g_1 + \eta_x g_2 + \eta_y g_3)dxdy.\end{aligned} \tag{4:9}$$

Again, the left-hand member may be written

$$\begin{aligned}\lim_{h\to 0}\Big\{\iint_S \eta(g_1{}^{(h)} - \frac{\partial}{\partial x} g_2{}^{(h)} - \frac{\partial}{\partial y} g_3{}^{(h)})dxdy \\ + \iint_S [\frac{\partial}{\partial x}(\eta g_2{}^{(h)}) + \frac{\partial}{\partial y}(\eta g_3{}^{(h)})]dxdy\Big\}.\end{aligned} \tag{4:10}$$

The second term vanishes as in (4:6). Owing to the equivalence of inequalities (4:7) and (4:8), and in view of the non-positiveness of η, the integrand in the first term of (4:10) is non-negative, and therefore the right-hand member of (4:9) is non-negative. But $\eta \equiv 0$ in R - S, and by the remarks following the definition of the class $\mathfrak{N}_S^m$ it is seen that $\eta_x = \eta_y = 0$ almost everywhere in R - S. Consequently, inequality (4:3) holds.

Suppose, secondly, that S' coincides wholly or in part with R'. As before, we propose to establish inequality (4:3), where η is a non-positive function in the class $\mathfrak{N}_S^n$. If, for $i = 1,2,\cdots$, we let $S_i = E[(x,y) \text{ in } S; \eta(x,y) < -1/i]$, and $S_0 = E[(x,y) \text{ in } S; \eta(x,y) < 0]$, then it is seen that $S - S_0 = E[(x,y) \text{ in } S; \eta(x,y) = 0]$, and $S_0 = \Sigma_i S_i$. Furthermore,

$S_i \subset S_{i+1}$ for each i, and $\lim\limits_{i\to\infty}$ meas.$(S_0 - S_i) = 0$. Now define functions η_i as follows:

$$\begin{aligned} \eta_i(x,y) &= \eta + \tfrac{1}{i} \text{ in } S_i, \\ \eta_i(x,y) &\equiv 0 \text{ in } R - S_i. \end{aligned} \tag{4:11}$$

Evidently, $\eta_i(x,y) = 0$ on S_i', and $\eta_i(x,y) < 0$ in S_i. It is also apparent that η_i is a.c.s. in R. The function η_i belongs to the class $\mathfrak{N}^n_{S_i}$ and hence, again by the discussion following the definition of the class $\mathfrak{N}^m_S$, we conclude that $\eta_{ix} = \eta_{iy} = 0$ almost everywhere in $R - S_i$. Moreover, $\eta_{ix} = \eta_x$ and $\eta_{iy} = \eta_y$ wherever η_x, η_y exist in S_i. These remarks, together with (4:11), enable us to write

$$\begin{aligned} &\iint\limits_R (\eta_i g_1 + \eta_{ix} g_2 + \eta_{iy} g_3)\,dxdy \\ &\quad = \frac{1}{i}\iint\limits_{S_i} g_1\,dxdy + \iint\limits_{S_0} (\eta g_1 + \eta_x g_2 + \eta_y g_3)\,dxdy \\ &\quad - \iint\limits_{S_0 - S_i} (\eta g_1 + \eta_x g_2 + \eta_y g_3)\,dxdy. \end{aligned} \tag{4:12}$$

Now the first integral of the right-hand member of (4:12) tends to zero as $i \to \infty$ since g_1 is summable in S_0; the limit of the third integral is also zero because $\lim\limits_{i\to\infty}$ meas. $(S_0 - S_i) = 0$. Furthermore, η belongs to $\mathfrak{N}^n_{S_0}$ by definition of the set S_0, and hence $\eta \equiv 0$ and $\eta_x = \eta_y = 0$ almost everywhere in $R - S_0$. Therefore, the second integral of the right-hand member of (4:12) is equal to the corresponding integral over R. It is possible of course for S_0 to be identical with R. We thus conclude, finally, that the limit of the integral of the left-hand side of (4:12) exists as $i \to \infty$ and is equal to the integral occurring in inequality (4:3). But each function $\eta_i(x,y)$ vanishes identically in a neighborhood of S_0', and so by the first part of the proof the left-hand member of (4:12) is non-negative for each i. This completes the proof.

By strengthening the hypotheses on g_1, g_2, g_3 in Lemma 4:2, the following important result may be deduced.

COROLLARY. If in Lemma 4:2 the function g_1 is continuous while g_2, g_3 are of class C' in R, then inequality (4:3) implies that

$$\frac{\partial}{\partial x} g_2 + \frac{\partial}{\partial y} g_3 - g_1 \geqq 0$$

in R. Conversely, the validity of this inequality implies (4:3) for every non-positive η in $\mathfrak{N}^n$.

The proof involves integration by parts and is basically like that of Lemma 4:2. The procedure, however, is now more direct, for it is no longer necessary to use integral means as approximating functions.

The foregoing results will now be applied in obtaining some of the key theorems of this paper. Let R be a bounded open set and let f(x,y,z,p,q) be a function defined and continuous for (x,y) in $\bar{R}$, and for (z,p,q) in an open region D of 3-dimensional space, not necessarily finite. The function f will be assumed to be of class $C^{(s)}$ in its last three arguments throughout D.

Presently a class of "admissible" functions, or surfaces, will be introduced. We shall be interested in establishing certain conditions that a specific admissible surface E: z = Z(x,y) furnish a relative minimum to the integral

$$I = \iint_R f(x,y,z(x,y),z_x(x,y),z_y(x,y))dxdy$$

in a prescribed class of surfaces z(x,y) which are dominated by Z(x,y) and coincide with Z(x,y) on the frontier of R. Obviously, we could just as well develop conditions that an admissible surface

Z furnish a relative minimum to the integral I in a corresponding class of surfaces *dominating* Z and coinciding with Z on R'. Conditions for this latter problem would be identical with those for the former except for changes in direction of certain inequalities. Hence, for simplicity, whenever there is occasion to refer to a condition of the latter sort, we shall give the reference to the corresponding condition for the problem actually under consideration.

Throughout this section, as well as subsequent sections, it will be understood, unless otherwise stated, that whenever we write, f, f_z, f_p, etc., the arguments to be assumed are $(x,y, Z(x,y), Z_x(x,y), Z_y(x,y))$.

DEFINITION 4:2. *A function* $z(x,y)$ *defined in* $\bar{R}$ *will be termed admissible of class* (m), *or simply admissible* (m) *in case:*

(i) *it is single valued and measurable in* $\bar{R}$;

(ii) $z_x(x,y)$, $z_y(x,y)$ *exist almost everywhere in* R *and are measurable in* R;

(iii) *for each* (x,y) *in* R *the point* $(z(x,y), z_x(x,y), z_y(x,y))$ *is in* D;

(iv) *the function* $f_z(x,y,z,z_x,z_y)$ *is summable in* R, *while* f_p, f_q *with the same arguments belong to the class* L^m;

(v) *the function* $f(x,y,z,z_x,z_y)$ *is summable in* R.

It is to be noted that the first three conditions imply measurability of f, f_z, f_p, f_q with arguments (x,y,z,z_x,z_y), and that the admissibility of z depends on the function f.

On account of (iv) in the above definition, we deduce as a special case of Lemma 4:2 the following theorem, which corresponds to the so-called Fundamental Lemma of the calculus of variations.

THEOREM 4:1. *Suppose that* E: $z = Z(x,y)$ *is admissible* (m) *and that for every non-positive* $\eta(x,y)$ *in the class* $\mathfrak{N}^n$ *the inequality*

$$(4{:}13) \qquad I_1[\eta] \equiv \iint_R (\eta f_z + \eta_x f_p + \eta_y f_q)\,dxdy \geqq 0$$

holds. Then, for almost all squares Q *in* R,

$$(4{:}14) \qquad \int_Q f_p dy - f_q dx \geqq \iint_T f_z dxdy.$$

Conversely, if for some Z *admissible* (m) *the relation* (4:14) *holds for almost all squares in* R, *then* (4:13) *is true for every non-positive function* η *in* $\mathfrak{N}^n$.

The expression appearing in (4:13) is of the form of the *first variation* of a double integral variation problem. The inequality (4:14) will be recognized as analogous to the integral form of the Haar differential equation for a double integral variation problem, and will frequently be referred to as the *Haar integral inequality*, or merely the *Haar inequality*.

If $Z(x,y)$ is of class $C^{(a)}$ in R, then f_z and the partial derivatives of f_p, f_q with respect to x and y are continuous in this region. Hence, by the corollary to Lemma 4:2, we deduce the following proposition.

COROLLARY. *If, in addition to satisfying the conditions of Theorem* 4:1, *the function* $Z(x,y)$ *is of class* $C^{(a)}$ *in* R, *then the non-negativeness of the first variation implies that*

$$\frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q - f_z \geqq 0$$

in R, *and conversely.*

The foregoing differential expression is the familiar Euler expression for a double integral variation problem. It should be recognized, in the following work, that whenever the function $Z(x,y)$ is of class $C^{(2)}$, then the above *Euler inequality* may be used in place of the Haar integral inequality.

We are now prepared to present some necessary conditions and some sufficient conditions that an admissible (m) function $Z(x,y)$ provide a relative minimum to the integral I with respect to unilateral variations. We shall concern ourselves with strong relative minima.

DEFINITION 4:3. *A function* $\eta(x,y)$ *is an admissible variation of class* m *or more simply, an admissible variation* (m) *in case:*

(i) η *belongs to the class* $\mathcal{N}^n$, *where, it will be recalled,* $n^{-1} = 1 - m^{-1}$;

(ii) *if* $z(x,y)$ *is an admissible surface* (m), *then for a sufficiently small in absolute value the point* $(z + a\eta,\ z_x + a\eta_x,\ z_y + a\eta_y)$ *is in* D *for* (x,y) *in* R, *and the function* $f(x,y,z + a\eta, z_x + a\eta_x, z_y + a\eta_y)$ *is summable in* R.

In particular, if η belongs to $\mathcal{N}^n_S$, where S is an open subset of R, η will be called an *admissible variation relative to* S. Given a $Z(x,y)$ admissible (m), we shall for convenience sometimes employ the phrase "corresponding admissible variation" to denote an admissible variation (m) associated with Z. Of particular interest is the case $m = n = 2$.

It will be noted that z is not required to be continuous in Definition 4:2. On the other hand, admissible variations must be continuous on account of (i) of Definition 4:3.

It is important to recognize that if Z is admissible (m) and η is a corresponding admissible variation, then it is not a consequence of our definitions that $Z + a\eta$ is an admissible surface (m) since in general condition (iv) of Definition 4:2 may not be satisfied along $Z + a\eta$. Although this latter property is in general missing, it seems more convenient for our present considerations to define admissible surface and admissible variation as above. Correspondingly, we shall speak of an admissible surface rendering a double integral a minimum with respect to corresponding admissible variations, rather than rendering the integral a minimum in a class of admissible surfaces.

It will be necessary for our purposes in the following paragraphs to distinguish between proper and improper minima. The admissible surface E: $z = Z(x,y)$ renders I an improper minimum in a class $\mathcal{E}$ of surfaces $z(x,y)$ in case

$$\iint_R f(x,y,Z,Z_x,Z_y)dxdy \leqq \iint_R f(x,y,z,z_x,z_y)dxdy$$

for every $z(x,y)$ in $\mathcal{E}$ not equivalent to $Z(x,y)$. For a proper minimum the equality sign is excluded.

Suppose that a function $Z(x,y)$ is admissible (m). Then we shall say for brevity that $Z(x,y)$ satisfies the condition (A^m) with respect to the function f and constant $d > 0$ [or, where there is no risk of confusion, merely, Z satisfies the condition (A^m)], in case whenever $|r| < d$, $|s| < d$, $|t| < d$, then $Z + r$, $Z_x + s$, $Z_y + t$ is in D, and moreover f_z, f_p, f_q with arguments $(x,y,Z+r,Z_x+s,Z_y+t)$ are all less in absolute value than a function $M(x,y)$ summable in R, and independent of r, s, t.

If $Z(x,y)$ is admissible (m) and Z_x, Z_y belong to L^m, we might note that condition (A^m) is valid if there exist constants K_1, K_2, and a function $M_0(x,y)$ summable in R, such that for $|r| < d$, $|s| < d$, $|t| < d$, the functions f_z, f_p, f_q with arguments $(x,y,Z+r,Z_x+s,Z_y+t)$ are all less in absolute value than

$$K_1|f(x,y,Z,Z_x,Z_y)| + K_2(|Z_x|^m + |Z_y|^n) + M_0(x,y).$$

For the two particular examples $f = p^2 + q^2$ and $f = (1+p^2+q^2)^{1/2}$ it may be shown that any function Z admissible (2) satisfies (A^2) with respect to f and any constant $d > 0$. In fact, for $f = p^2+q^2$ one might take $K_1 + K_2 = 1$ and $M_0 = d^2 + 2$; for $f = (1+p^2+q^2)^{1/2}$ the values $K_1 = K_2 = 0$, $M_0 = 1$ are suitable.

THEOREM 4:2. *Suppose* E: $z = Z(x,y)$ *is an admissible surface* (m) *satisfying condition* (A^m), *and that* E *furnishes a relative minimum to the integral* I *with respect to non-positive admissible variations* η. *Then*: (i) $I_1[\eta] \geqq 0$; (ii) *for almost all squares in* R *the Haar inequality* (4:14) *holds.*

We shall commence by proving (i) for Lipschitzian functions $\eta(x,y)$. This limitation is to be removed later. By hypothesis, if a is non-negative and less than some positive constant $1/k'$ where $k' > 1$, then

$$(4:15) \quad \iint_R f(x,y,Z,Z_x,Z_y)dxdy \leqq \iint_R f(x,y,Z+a\eta,Z_x+a\eta_x,Z_y+a\eta_y)dxdy.$$

We note that for $a = 0$, the equality holds. The partial derivative of the integrand on the right with respect to the parameter a exists and is expressed by

$$\frac{\partial}{\partial a} f(x,y,Z+a\eta,Z_x+a\eta_x,Z_y+a\eta_y) = \eta f_z + \eta_x f_p + \eta_y f_q,$$

the arguments of f_z, f_p, f_q being the same as those of f on the left. Suppose that k'' is an upper bound for $|\eta|$, and that k''' is the Lipschitzian constant corresponding to η. Then let $k = \max\{dk', k'', k'''\}$, d being the number appearing in condition (A^m), and keep $|a| < d/k$. Then $a\eta$, $a\eta_x$, $a\eta_y$ are all less in absolute value than d. As a consequence of condition (A^m) we therefore conclude that

$$\left|\frac{\partial}{\partial a} f(x,y,Z+a\eta,Z_x+a\eta_x,Z_y+a\eta_y)\right| \leqq 3dM(x,y), \quad (|a| < d/k)$$

in S. One may accordingly differentiate in (4:15) under the integral sign whenever $|a| < d/k$. In particular, inasmuch as the equality holds in (4:15) when $a = 0$, we have $I_1[\eta] \geqq 0$, which is the desired conclusion (i) for Lipschitzian η's. The reader will remember at this juncture that the first part of Theorem 4:1 was proved merely under the assumption that η satisfy a Lipschitz condition. Statement (ii) of the present theorem is a direct consequence of that result. But again, by Theorem 4:1, it follows that statement (ii) implies (i) with the restriction on the character of η removed.

In the foregoing two theorems it was not necessary to assume the function $Z(x,y)$ continuous. However, continuity of Z is essential in the proof of the following result.

THEOREM 4:3. *Suppose that* E: $z = Z(x,y)$ *is continuous and admissible* (m), *and that* E *renders the integral* I *a proper relative minimum in the class of functions* $z(x,y)$ *such that* $z = Z$ *on* R', *and* $Z - d \leqq z \equiv Z + \eta \leqq Z$ *in* R, η *being an admissible variation* (m). *Let* S *be any subregion of* R *for which there exists a continuous function* $W(x,y)$ *that is admissible* (m), *coincides with* Z *on* S', *and has the following properties:*

(i) W *renders the integral* $\iint_S f\,dx\,dy$ *a relative minimum with respect to all variations which are admissible* (m) *and not greater in absolute value than* d;

(ii) $|Z - W| \leqq d$ *in* S;

(iii) *the function defined to be equal to* $Z - W$ *in* $\bar{S}$ *and identically zero in* $\bar{R} - S$ *is an admissible variation* (m).

Then $Z \leqq W$ *in* S.

This theorem could be strengthened inasmuch as all the implications of admissibility are not utilized in the proof. However, the usefulness of this theorem in following sections will be in connection with absolute minima of double integrals. It will be noted in Section 5 for example, that under circumstances there prescribed a surface which is admissible (m) and furnishes a relative minimum to the integral I, furnishes, in actuality, an absolute minimum to I. Hence, the present theorem will be more effective there if stated for admissible functions and variations. It states in essence that a function Z(x,y) minimizing the integral I with respect to non-positive variations is dominated by any function W(x,y) which coincides with Z on the boundary of R and which furnishes I with a relative minimum with respect to general variations, provided that Z(x,y) lie close enough to W(x,y).

If the theorem be false, then there must exist a point (x_0,y_0) in R at which $Z(x_0,y_0) > W(x_0,y_0)$, and hence an open subset N of R containing (x_0,y_0) in which $Z > W$ and on the frontier of which $Z = W$. Then, since $|Z - W| \leqq d$ in R, we have, by virtue of (iii) and the hypothesized minimizing property of Z(x,y),

$$(4:16) \qquad \iint_N f(x,y,Z,Z_x,Z_y)dxdy < \iint_N f(x,y,W,W_x,W_y)dxdy.$$

On the other hand, the minimizing property of W(x,y) implies the contradictory statement,

$$(4:17) \qquad \iint_N f(x,y,Z,Z_x,Z_y)dxdy \geqq \iint_N f(x,y,W,W_x,W_y)dxdy.$$

Accordingly, the assumption that $Z > W$ at some point of R must be rejected. It should be noted that the conclusion of the theorem would also hold if it had been required that W furnish a proper relative minimum while Z be permitted to furnish but an improper relative minimum. The contradiction in the proof would then arise from the fact that the equality sign would be admitted in (4:16) but not in (4:17).

The Weierstrass excess function is defined to be

$$\mathcal{E}(x,y,z,p,q;P,Q) = f(x,y,z,P,Q) - f(x,y,z,p,q) - (P - p)f_p(x,y,z,p,q) - (Q - q)f_q(x,y,z,p,q).$$

Suppose that admissible surfaces and variations are required to be piecewise of class C'. For this case L. M. Graves [2] has produced a proof of the Weierstrass condition for multiple integral problems which may be applied with almost no alteration to the unilateral case. It is to be remarked at this point that under the conditions of this paragraph, if Z is an admissible surface, and if η is an admissible variation, then $Z + a\eta$ is an admissible surface for a sufficiently small. The Weierstrass condition is as follows:

THEOREM 4:4. *If* E: $z = Z(x,y)$ *is piecewise of class* C' *and minimizes* I *with respect to unilateral variations, then at each element* $(x,y,Z(x,y),Z_x(x,y),Z_y(x,y))$ *of* E *the inequality* $\mathcal{E}(x,y,Z,Z_x,Z_y;P,Q) \geqq 0$ *holds for every pair* P, Q *such that* (Z, P, Q) *is in* D.

From this, by the usual application of Taylor's theorem we derive the Legendre condition:

THEOREM 4:5. *Under the hypotheses of Theorem* 4:4, *at each element* (x,y,Z,Z_x,Z_y) *of* E *the quadratic form*

$$A^2f_{pp} + 2ABf_{pq} + B^2f_{qq} \tag{4:18}$$

in (A,B) *is positive semi-definite.*

A sufficiency proof by an expansion method applicable to the present problem has been developed by W. T. Reid [12] in a paper to be published soon. For simplicity, Reid's notation will be closely followed in submitting the corresponding theorem for the unilateral case.

In the remainder of this section, we shall understand that all admissible functions, as well as admissible variations, are to be Lipschitzian in $\bar{R}$. Here again, it is noted that if Z is an admissible surface and if η is an admissible variation, then the surface $Z + a\eta$ is admissible for a sufficiently small. A surface $z = z(x,y)$ is called non-singular if $f_{pp}f_{qq} - f_{pq}^2 \neq 0$ at all elements $(x,y,z(x,y),z_x(x,y),z_y(x,y))$ belonging to the surface. A function $u(x,y)$ which is continuous in $\bar{R}$, of class C' in R, and whose partial derivatives u_x, u_y coincide with functions continuous in $\bar{R}$, will be said to satisfy the integral form of the Haar accessory equation if

$$\int_Q \omega_\pi(x,y,u,u_x,u_y)dy - \omega_\kappa(x,y,u,u_x,u_y)dx = \iint_T \omega_\zeta(x,y,u,u_x,u_y)dxdy$$

holds for all squares Q of R, where

$$2\omega(x,y,\zeta,\pi,\kappa) = f_{pp}\pi^2 + 2f_{pq}\pi\kappa + f_{qq}\kappa^2 + 2f_{pz}\pi\zeta + 2f_{qz}\kappa\zeta + f_{zz}\zeta^2.$$

We shall say that a function $Z(x,y)$ satisfies the condition IV^* if their exists a solution of the kind described above of the Haar accessory equation which is nowhere zero in $\bar{R}$. Moreover, E is said to satisfy condition II_N^* in case there exists in (x,y,z,p,q)-space a one-sided neighborhood N of the elements (x,y,Z,Z_x,Z_y) of the form N: (x,y) in $\bar{R}$, $|p - Z_x(x,y)| \leqq a_1$, $|q - Z_y(x,y)| \leqq a_2$, $Z(x,y) - a_3 \leqq z \leqq Z(x,y)$, such that for (x,y,z,p,q) in N, and $(P,Q) \neq (p,q)$, we have $\mathcal{E}(x,y,z,p,q;P,Q) > 0$. It is clear that the validity of condition II_N^* for E, together with the non-singularity of E, implies the strengthened form of the Legendre condition; that is, the quadratic form (4:18) is positive definite.

The proof of Reid's theorem depends on two preliminary lemmas which will not be stated here. The theorem for the unilateral case is as follows:

THEOREM 4:6. *If* E: $z = Z(x,y)$ *is a non-singular admissible surface that is continuous in* $\bar{R}$, *of class* C' *in* R, *which satisfies the Haar inequality, and also conditions* II_N^* *and* IV^*, *then* E *renders* I *a strong relative minimum with respect to non-positive admissible variations.*

5. *A class of positive regular problems.* As a special case of the problems treated in the foregoing section, we now turn to integrand functions $f(x,y,p,q)$ which are continuous in all four arguments, and of class $C^{(2)}$ for (x,y) in $\bar{R}$ and p, q arbitrary. The problem will be assumed to be *positive regular*; i.e., the quadratic form

$$f_{pp}A^2 + 2f_{pq}AB + f_{qq}B^2 \tag{5:1}$$

is positive definite for all (x,y,p,q) in the region of definition of f. All the theorems of the last section may, of course, be applied to the present situation. But, owing to the special nature of the integrand function, it is natural to expect that the results to be obtained here would be somewhat more individualized. The most conspicuous phenomena to be observed in the sequel are the identically vanishing of the partial derivative f_z, and the fact that the results provide for absolute minima. The latter feature is a consequence of the hypothesis of positive regularity. Since $f_z \equiv 0$, (4:13) reduces to

$$I_1[\eta] = \iint_R (\eta_x f_p + \eta_y f_q)\,dxdy \geq 0,$$

while the Haar inequality (4:14) becomes

$$\int_Q f_p dy - f_q dx \geq 0.$$

Outside of these changes, the statement of Theorem 4:1 remains the same for the case considered here. Theorems 4:2 and 4:3 will be interpreted in the light of the hypothesis of positive regularity. Theorem 5:1, which corresponds to Theorem 4:6, is here proved without the qualification previously demanded that the surface be continuous in $\bar{R}$ and of class C' in R. Theorems 5:2 and 5:3 have no counterpart in the preceding treatment. They are proved only for functions $f(p,q)$, not involving x, y, z explicitly. In Section 6 corresponding results will be proved under weaker hypotheses for the particular case where $f(p,q) = p^2 + q^2$, the integrand of the Dirichlet integral.

By applying Taylor's theorem with remainder to the Weierstrass $\mathcal{E}$ -function for this problem, we find in view of the positive definiteness of the quadratic form (5:1) that the $\mathcal{E}$-function is positive for $(P,Q) \neq (p,q)$ and the strengthened condition of Legendre is satisfied throughout the region under consideration. We can now derive a stronger result than Theorem 4:6. For, suppose that the Haar inequality holds for $Z(x,y)$ admissible (m), and let $z(x,y) = Z(x,y) + \eta(x,y)$ be any other surface such that η is a non-positive variation admissible (m) relative to R. Then, by Taylor's theorem with remainder,

$$
\begin{aligned}
I &\equiv \iint\limits_R f(x,y,z_x,z_y)\,dxdy - \iint\limits_R f(x,y,Z_x,Z_y)\,dxdy \\
&= \iint\limits_R (\eta_x f_p(x,y,Z_x,Z_y) + \eta_y f_q(x,y,Z_x,Z_y))\,dxdy \qquad (5{:}2)\\
&\quad + \frac{1}{2}\iint\limits_R (\eta_x^2\,\bar{f}_{pp} + 2\eta_x\eta_y\bar{f}_{pq} + \eta_y^2\,\bar{f}_{qq})\,dxdy,
\end{aligned}
$$

where $\bar{f}_{pp}$, etc., signify $f_{pp}(x,y,Z_x+\theta\eta_x,Z_y+\theta\eta_y)$, etc., and $0 < \theta(x,y) < 1$. It is to be noted that the integral of the quadratic expression on the right is finite in consequence of the expansion of $f(x,y,z_x,z_y)$ by Taylor's theorem with remainder, and the summability of $f(x,y,z_x,z_y)$, $f(x,y,Z_x,Z_y)$, and $\eta_x f_p + \eta_y f_q$.

By virtue of Theorem 4:1 and the positive regularity of the problem it then follows that $\Delta I > 0$ for $\eta \not\equiv 0$ in R. We have thus proved the following result.

THEOREM 5:1. *Under the conditions imposed above, if the Haar inequality is satisfied for* Z(x,y) *an admissible function* (m), *then* Z(x,y) *renders the integral* I *a proper absolute minimum with respect to corresponding non-positive admissible variations.*

It results from this theorem and Theorem 4:2 that if an admissible (m) function Z furnishes a *relative* minimum to I then it actually furnishes an *absolute* minimum. Hence, for the type of multiple integrals considered in this section, Theorems 4:2 and 4:3 are not rendered weaker by supposing that Z furnishes an absolute minimum to the integral I.

A type of function employed in the discussion of surfaces minimizing an integral $\iint f(p,q)dxdy$ is the so-called *saddle-function* [9, pp. 62,63]. The definition of such a function depends on the following definition of monotonicity for a function of two variables.

DEFINITION 5:1. *A function* g(x,y) *continuous in* $\bar{R}$ *is monotonic in* R *in case for every connected open set* S *contained in* R, *the function* g(x,y) *attains the maximum and the minimum of its values in* $\bar{S}$ *on the frontier* S'.

DEFINITION 5:2. *If the function* g(x,y) - (ax + by + c) *is monotonic in* R *for every choice of the constants* a, b, c, *then* g(x,y) *is called a saddle-function in* R.

It may be shown that under prescribed conditions a continuous function minimizing the integral I with respect to unrestricted variations is a saddle-function [9, pp. 65, 66]. In our problem, we can prove that *an admissible* (m) *function* Z(x,y) *continuous in* $\bar{R}$, a.c.s. *in* R, *whose first order* *partial derivatives belong to* L^n, *and which minimizes* I *with respect to non-positive admissible*(m) *variations is what might be called an upper semi-saddle function*; i.e., it is a function such that for every choice of constants a, b, c, and for every open subregion S in R, the function $Z(x,y) - (ax + by + c)$ attains the maximum of its values in $\overline{S}$ on the frontier S'. For, suppose that $Z(x,y)$ is not such a function. Then there exist constants a, b, c and a region S in R such that for some point (x_0,y_0) in S we have $Z(x_0,y_0) - (ax_0 + by_0 + c) > M'$, where M' is the maximum of $Z(x,y) - (ax + by + c)$ on S'. Hence, letting $z(x,y) = ax + by + c + M'$, there exists a connected open subset N of S in which $z(x,y) - Z(x,y) < 0$ and on the frontier of which $z - Z = 0$. Moreover, in view of our hypotheses on Z, the function $z - Z$ is an admissible variation (m) relative to N. Now, it is easily shown that the plane $z(x,y)$ renders the integral I an absolute minimum with respect to unrestricted variations, and hence, by Theorem 4:3 and the remarks made in this section on that theorem, we conclude that $Z(x,y) \leqq ax + by + c + M'$ in N. This, however, contradicts our assumption on the character of Z. Geometrically, the property we have proved amounts to saying that there are no hillocks or humps on the surface; there can be only cavities or pockets.

In the remainder of this section we shall assume that $f = f(p,q)$; i.e., the integrand function does not involve x, y, z explicitly. In addition to the conditions imposed at the beginning of this section, we shall also suppose that the second derivatives of f satisfy a uniform Hölder condition in each bounded portion of (p,q)-space. Consider the circle $C(x,y;r)$ with arclength s, $0 \leqq s \leqq 2\pi r$, and let $z(s)$ be of class $C^{(2)}$ in the closed interval $(0,2\pi r)$. Then $z(s)$ defines a space curve C whose projection on the xy-plane is $C(x,y;r)$; moreover, as will be proved at the end of this section, $z(s)$ satisfies the *three-point condition* [9, p. 49]. That is, if P, P_1, P_2 are any three distinct points on C, and if $\Phi(P,P_1,P_2)$ denote the positive acute angle between the z-axis and the normal to the plane determined by P, P_1, P_2 then $\cos \Phi$ is bounded away from zero for all possible distinct positions of P, P_1, P_2 on C. Hence, by an existence theorem of Haar [4, p. 19], there exists a Lipschitzian function $W(x,y)$ defined in $\bar{D}(x,y;r)$ which takes on the values $z(s)$ on $C(x,y;r)$ and which minimizes the integral I in the class of all Lipschitzian functions $z(x,y)$ having the values $z(s)$ on $C(x,y;r)$. Haar's theorem really applies to a broader situation than the one proposed here, but we shall not need his theorem in all its generality. It then turns out, by a theorem due to Morrey [7, pp. 158-160], that the minimizing function $W(x,y)$ is of class $C^{(2)}$, and the second order partial derivatives of $W(x,y)$ satisfy a Hölder condition in each closed subregion of $D(x,y;r)$.

THEOREM 5:2. *Suppose* $f(p,q)$ *is positive regular and satisfies the differentiability conditions described above. Suppose, moreover, that* $Z(x,y)$ *is admissible* (m), *continuous in* $\bar{R}$, *of class* $C^{(2)}$ *in* R, *and that in every disc* $\bar{D}(x,y;r)$ *contained in* R, $Z(x,y)$ *is dominated by the minimizing function* $W(x,y)$ *which coincides with* Z *on* $C(x,y;r)$. *Then* $I_1[\eta] \geqq 0$ *for every non-positive* η *in the class* $\mathcal{N}^n$.

Suppose first that there exists an open set S, comprised with its frontier in R, and an η in $\mathcal{N}^n_S$ such that $I_1[\eta] < 0$. Then, by a procedure now familiar, we find that

$$\iint_R \eta \left(\frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q\right)dxdy > 0.$$

In view of our hypothesis on the differentiability of $Z(x,y)$ the partial derivatives written exist and are continuous. Hence, there is an open subset A of R on which $\partial f_p/\partial x + \partial f_q/\partial y < 0$. Select a circle $C(x_0,y_0;r)$ contained with its interior $D(x_0,y_0;r)$ in A. By the corollary to Theorem 4:1, it is seen that

$$\iint_{D(x_0,y_0;r)} (\zeta_x f_p + \zeta_y f_q)\,dxdy > 0$$

for every ζ non-negative and not equivalent to zero in $\mathfrak{M}^n_{D(x_0,y_0;r)}$. Then, by (5:2) the function Z renders the integral

$$\iint_{D(x_0,y_0;r)} f(z_x,z_y)\,dxdy$$

a proper absolute minimum with respect to corresponding non-negative admissible variations. Consequently, by Theorem 4:3, by the comments made in this section on that theorem, and by the above remarks on the existence of minimizing functions, there exists a function W coinciding with Z on $C(x_0,y_0;r)$, minimizing the integral with respect to admissible variations (m) of arbitrary sign, and dominated by Z in $D(x_0,y_0;r)$; furthermore $Z \not\equiv W$ there. For if it were true that $Z \equiv W$ in $D(x_0,y_0;r)$, then the Euler equation, $\partial f_p/\partial x + \partial f_q/\partial y = 0$, would hold along Z in $D(x_0,y_0;r)$, and this is contrary to the inequality deduced for the Euler expression in A. But then the hypothesis that $Z \leqq W$ in $D(x_0,y_0;r)$ is contradicted. And now, to prove the required result for the case where S' coincides wholly or in part with R', one may proceed as in the proof of Lemma 4:2.

From the foregoing results we deduce immediately the following theorem.

THEOREM 5:3. *If the function* $Z(x,y)$ *satisfies the conditions of Theorem* 5:2, *then it renders the integral* I *a proper absolute minimum with respect to corresponding non-positive* admissible variations.

It is to be remarked that for integrand function $f(p,q)$ and Lipschitzian functions $Z(x,y)$ one may, by the "method of variation of the independent variable" employed by Haar [3, pp. 144-146] and Radó [8], deduce inequalities analogous to certain equations which are auxiliary to the Haar equation. For the case of the Haar equation these auxiliary equations are of paramount importance. For the Haar inequality, however, the author has been unable to find significant consequences of the associated inequalities, and for brevity they will be omitted here.

We shall now digress briefly into the realm of differential geometry in order to establish the validity of the three-point condition under the assumptions made on page 30 and to prove that the curve there described has an osculating plane at each point.

LEMMA 5:1. On the circle $C(x,y;r)$ with arclength s, let $z(s)$ be a periodic function of class $C^{(a)}$, with period $2\pi r$. Then $z(s)$ defines a space curve Γ satisfying the three-point condition and having an osculating plane at each point.

It obviously suffices to consider the case where $C(x,y;r)$ is the unit circle $C = C(0,0;1)$ in which event the parametric equations of the space curve may be written

$$\Gamma \quad : \; x = \cos\theta, \qquad y = \sin\theta, \qquad z = z(\theta),$$

where $z(\theta)$ is of class $C^{(a)}$ and is periodic of period 2π. Let P, P_1, and P_2 be three distinct points on Γ. As the angle $\Phi(P,P_1,P_2)$ defined on page 31 remains invariant under rotation of the x,y axes, we may always assume that these axes are so rotated that the x-axis passes through the projection of P on C, and P corresponds to $\theta = 0$. Let θ_1 and θ_2 be parameter values determining P_1 and P_2, respectively. The equation of the plane through P, P_1, P_2 is then

$$\begin{vmatrix} X - 1 & Y & Z - z(0) \\ \cos\theta_1 - 1 & \sin\theta_1 & z(\theta_1) - z(0) \\ \cos\theta_2 - 1 & \sin\theta_2 & z(\theta_2) - z(0) \end{vmatrix} = 0$$

Letting $a(\theta_1,\theta_2)$, $b(\theta_1,\theta_2)$, $c(\theta_1,\theta_2)$ denote, respectively, the cofactors of the elements in the first row of the determinant, we have

$$\cos\Phi(P,P_1,P_2) = \frac{c}{(a^2 + b^2 + c^2)^{1/2}}\,. \tag{5:3}$$

It is easily shown that

$$c = 4 \sin\frac{\theta_2 - \theta_1}{2} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2}.$$

We now proceed to prove that (5:3) may be written

$$\cos\Phi(P,P_1,P_2) = c'(\theta_1,\theta_2)/k(\theta_1,\theta_2), \tag{5:4}$$

where $k(\theta_1,\theta_2)$ is bounded and

$$c'(\theta_1,\theta_2) = \left[\sin\frac{\theta_2 - \theta_1}{2}\Big/\frac{(\theta_2 - \theta_1)}{2}\right]\left[\sin\frac{\theta_1}{2}\Big/\frac{\theta_1}{2}\right]\left[\sin\frac{\theta_2}{2}\Big/\frac{\theta_2}{2}\right].$$

Using the Maclaurin expansion for $\cos\theta$, and defining

$$v(\theta) \equiv \frac{z(\theta) - z(0)}{\theta}, \qquad v(0) \equiv \lim_{\theta\to 0} \frac{z(\theta) - z(0)}{\theta} = z'(0),$$

we may obtain, after suitable manipulations on the rows of b,

$$b = 1/2\ (\theta_2 - \theta_1)\theta_1\theta_2 b'(\theta_1,\theta_2),$$

where

$$b'(\theta_1,\theta_2) = -\begin{vmatrix} -\theta_1 + g(\theta_1) & \dfrac{z(\theta_1) - z(0)}{\theta_1} \\ -1 + h(\theta_1,\theta_2) & \dfrac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} \end{vmatrix}$$

In this determinant, $g(\theta_1)$ and $h(\theta_1,\theta_2)$ are convergent power series containing no terms of degree lower than the second. We proceed in the same manner to show that

$$a = 1/2\ (\theta_2 - \theta_1)\theta_1\theta_2 a'(\theta_1,\theta_2),$$

where

$$a'(\theta_1,\theta_2) = \begin{vmatrix} 2 + g_0(\theta_1) & \dfrac{z(\theta_1) - z(0)}{\theta_1} \\ h_0(\theta_1,\theta_2) & \dfrac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} \end{vmatrix},$$

and g_0, h_0 are convergent power series containing no terms of degree lower than the first.

We shall be certain that $a'(\theta_1,\theta_2)$ and $b'(\theta_1,\theta_2)$ are uniformly bounded as soon as the boundedness of $|v(\theta_2) - v(\theta_1)|/(\theta_2 - \theta_1)$ has been verified. Now $v(\theta)$ is continuous and may be shown to possess a derivative everywhere; in fact, for $\theta \neq 0$, we have,

$$v'(\theta) = \frac{\theta z'(\theta) - z(\theta) + z(0)}{\theta^2}.$$

Expressing $z(0)$ by Taylor's expansion with remainder about the point θ, we find this equality to be equivalent to

$$v'(\theta) = z''(\bar{\theta})/2, \qquad 0 < |\bar{\theta}| < |\theta|.$$

Also, by the definition of $v(\theta)$, we obtain

$$\frac{v(\theta) - v(0)}{\theta} = \frac{z(\theta) - z(0) - \theta z'(0)}{\theta^2}.$$

Again expanding $z(\theta)$, this time about the point $\theta = 0$, it is seen that

$$\frac{v(\theta) - v(0)}{\theta} = z''(\theta^*)/2, \qquad 0 < |\theta^*| < |\theta|.$$

Therefore, by the continuity of z'', $v'(0)$ exists and is equal to $z''(0)/2$. Hence $v'(\theta)$ exists everywhere on the given circle and is uniformly bounded by the finite upper bound of $z''(\theta)$. Then by the mean value theorem, $[v(\theta_2) - v(\theta_1)]/(\theta_2 - \theta_1) = v'(\bar{\theta})$, where $\bar{\theta}$ is an intermediate value of θ, so that a' and b' are clearly bounded. This result, together with (5:3), implies (5:4) where $k(\theta_1,\theta_2) = (a'^2 + b'^2 + c'^2)^{1/2}$ is bounded. It should be remarked that a', b', c' are proportional to the direction cosines of the normal to the plane through P, P_1, P_2.

Let e be a small positive number, and restrict θ_1, θ_2 to be in the closed interval $(-\pi, \pi - e)$. Then none of the three factors of $c'(\theta_1,\theta_2)$ can become zero; indeed, each factor assumes a positive minimum. Consequently, since $k(\theta_1,\theta_2)$ is bounded, the product in (5:4) is bounded away from zero. Let the greatest lower bound of $\cos \Phi(P,P_1,P_2)$ be denoted by m_1, for θ_1, θ_2 restricted as above. Let m_2 and m_3 have similar meanings corresponding to the respective closed intervals $(-\pi + e, \pi)$ and $(\pi - e, \pi + e)$. Every possible triple of distinct points P, P_1, P_2 on Γ corresponds to a pair of values θ_1, θ_2 lying in at least one of the three intervals indicated above. Hence, for all possible triples of distinct points P, P_1, P_2, the function $\cos \Phi(P,P_1,P_2)$ is bounded away from zero by the smallest of the numbers m_1, m_2, m_3, and the three-point condition is therefore satisfied.

The quantities θ_1, θ_2 will now be confined to an interval $(-e,e)$, where e is small, and it will be proved that the direction numbers $a'(\theta_1,\theta_2)$, $b'(\theta_1,\theta_2)$, $c'(\theta_1,\theta_2)$ approach limits as P_1, P_2 approach P independently along Γ. Now in view of the properties of $v(\theta)$ inferred in the foregoing proof, we see that

$$\frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} = v'(\bar{\theta}) = z''(\theta^*)/2,$$

where $\bar{\theta} = \theta_1 + a(\theta_2 - \theta_1)$ for some $0 < a < 1$, and $0 < |\theta^*| < |\bar{\theta}|$. Then $|\theta^*| \leqq \max\left\{|\theta_1|, |\theta_2|\right\}$. Hence, it is easily seen from the continuity of z'' that

$$\lim_{\substack{\theta_1 \to 0 \\ \theta_2 \to 0}} \frac{v(\theta_2) - v(\theta_1)}{\theta_2 - \theta_1} = z''(0)/2.$$

It now follows readily that

$$\lim_{\substack{\theta_1 \to 0 \\ \theta_2 \to 0}} a'(\theta_1,\theta_2) = z''(0), \lim_{\substack{\theta_1 \to 0 \\ \theta_2 \to 0}} b'(\theta_1,\theta_2) = z'(0), \lim_{\substack{\theta_1 \to 0 \\ \theta_2 \to 0}} c'(\theta_1,\theta_2) = 1.$$

Thus, the plane determined by the points P, P_1, P_2 approaches a limiting position as P_1, P_2 approach P along the curve, and the limiting plane is by definition the osculating plane to Γ at P. But P is an arbitrary point on Γ, so that we have proved that Γ has an osculating plane at each point.

6. Subharmonic functions. In this section we shall consider, as a particular case of the class of integrand functions treated in Section 5, the function $f(p,q) = p^2 + q^2$, the integrand of the Dirichlet integral. The problem treated here is clearly positive regular. If $Z(x,y)$ is single valued and measurable in $\bar{R}$, and if the partial derivatives Z_x, Z_y exist almost everywhere and belong to L^2 in R, then Z is admissible (2). The class of corresponding admissible variations in this case is composed of functions $\eta(x,y)$, continuous in $\bar{R}$, a.c.s. in R, vanishing on R', and whose first order partial derivatives belong to L^2 in R. This section will be concerned partly, but not entirely, with functions and variations which are admissible (2); for simplicity, in this section we shall say "Z is admissible" instead of the more exact phrase "Z is admissible (2)".

Before undertaking the subject of subharmonic functions it will be necessary to make a few preliminary assertions regarding harmonic functions. Let $U(x,y)$ be harmonic on an open set S and continuous in $\bar{S}$. Then $U(x,y)$ satisfies the Haar integral equation; i.e.,

$$\int_C U_x dy - U_y dx = 0$$

for every simply closed rectifiable curve C contained with its interior in S. McShane [6] has proved that U renders the Dirichlet integral

$$I[z;S] = \iint_S (z_x^2 + z_y^2)dxdy$$

an absolute minimum in the class of functions $z(x,y)$ coinciding with U on S', continuous in $\bar{S}$, and a.c.s. in S. The following possibility should be observed. We may have $I[U;S] = \infty$; in such a case, on account of the minimizing property of U, the class of *admissible* functions continuous in $\bar{S}$, a.c.s. in S, and coinciding with $U(x,y)$ on S' will be vacuous. We shall come back to this point later in our discussion.

We shall employ the following definition of subharmonic function which, except for the final statement, is the same as that given by Radó [11, p. 1].

DEFINITION 6:1. *A function* $u(x,y)$ *defined in a connected region* R *is subharmonic in* R *if it satisfies the following conditions:*

(i) $-\infty \leq u < +\infty$ *in* R;

(ii) u *is not identically equal to* $-\infty$ *in* R;

(iii) u *is upper semi-continuous in* R;

(iv) *if* S *is any open region such that* $\bar{S}$ *is contained in* R, *and if* $U(x,y)$ *is harmonic in* S, *continuous in* $\bar{S}$, *and* $U \geq u$ *on*

S', _then_ U $\geqq$ u _in_ S.

If the region R _is not connected, then_ u(x,y) _is subharmonic in_ R _in case it is subharmonic in every connected subregion of_ R.

A function u which is subharmonic in R is summable on every measurable set comprised with its frontier in R[11, p.4]. Furthermore, u is summable as a function of the polar angle on every circle $C(x_0,y_0;r)$ contained with its interior in R[11, p. 4]. A fundamental property of subharmonic functions is that

$$u(x_0,y_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta,\ y_0 + r\sin\theta)d\theta$$

for every (x_0,y_0) in R and r such that $\bar{D}(x_0,y_0;r)$ is in R [11, pp. 3,4]. Conversely, a function satisfying this inequality in R and also (i), (ii) and (iii) of Definition 6:1 is certainly subharmonic in R [11, p. 7]. Another important inequality satisfied by subharmonic functions, and derived from the above, is

$$u(x_0,y_0) \leqq \frac{1}{\pi r^2}\int_0^{2\pi}\int_0^{r} u(x_0 + \rho\cos\theta,\ y_0 + \rho\sin\theta)\rho\, d\rho\, d\theta,$$

for every (x_0,y_0) in R and r such that $\bar{D}(x_0,y_0;r)$ is in R [11, pp. 3,4]. Again, the validity of this inequality in R, together with (i), (ii) and (iii) of Definition 6:1, is sufficient to characterize a function as subharmonic in R [11, p. 7].

F. Riesz has shown [13, part II] that if u(x,y) is an arbitrary subharmonic function in the region R, then in any open set G contained with its frontier in R, u may be represented as the sum of the potential of a negative mass distribution and a harmonic function. A simplified development of this representation due to G. C. Evans [1, part I, p. 237] is reviewed by Radó [11, pp. 40-42]. In consequence of this result, it may be shown [11, p. 45] that in G the subharmonic function u is an absolutely continuous function of y for almost every x, and an absolutely continuous function of x for almost every y; moreover, its partial derivatives are summable in G. It is to be observed that u need not be admissible, so that the possibility exists that $I[u;R] = \infty$.

It is well known that the function defined at each point to be equal to the integral mean of a subharmonic function over a circular disc of fixed radius is also subharmonic within its range of definition [11, pp. 10, 11]. But this property is also valid if the integral means be taken over square regions [10, p. 347]; thus the function $u^{(h)}(x,y)$ is subharmonic in its region of definition N_h. And now, by the remarks of the preceding paragraph and Property 3:7 we have that

$$(6{:}1) \qquad \frac{\partial}{\partial x} u^{(h)} = \left(\frac{\partial u}{\partial x}\right)^{(h)},$$

$$(6{:}2) \qquad \frac{\partial}{\partial y} u^{(h)} = \left(\frac{\partial u}{\partial y}\right)^{(h)}$$

in N_h. Therefore, if u is subharmonic in R, then $u^{(h)}$ is of class C' in N_h.

Having made the foregoing observations, we are now prepared to present a number of propositions which correspond to theorems and properties already obtained in Sections 4 and 5. We shall restrict ourselves, for the time being, to admissible functions and variations. We notice that for admissible functions and variations $w(x,y)$ and $\eta(x,y)$, respectively, the first variation is

$$I_1[\eta] = \iint_R (\eta_x w_x + \eta_y w_y)\,dxdy,$$

while the Haar inequality is

$$\int_Q w_x dy - w_y dx \geqq 0.$$

Corresponding to Theorem 4:1 we have that *if* $I_1[\eta] \geqq 0$ *for all non-positive admissible variations, then the Haar inequality holds for almost all squares of* R, *and conversely*. Inasmuch as the proposition corresponding to Theorem 5:1 is to be referred to later on in the proof of a somewhat stronger theorem, we shall write it as a lemma.

LEMMA 6:1. *The non-negativeness of the first variation, or the validity of the Haar inequality for almost all squares of* R, *implies that an admissible* w *affords the Dirichlet integral a proper minimum in the class of functions* $w + \eta$, *where* η *is a non-positive admissible variation.*

We may also apply Theorem 4:3 to our present considerations. However, in view of remarks made in Section 5 on that theorem, its statement may be simplified. We again begin with *an admissible surface* w(x,y) *which minimizes the Dirichlet integral with respect to non-positive admissible variations. Suppose, furthermore, that* w(x,y) *is continuous in* $\bar{R}$ *and* a.c.s. *in* R. *Then, if* S *is any open subset of* R *for which there exists a function* U(x,y) *harmonic in* S, *continuous in* $\bar{S}$, *and* $U \geqq w$ *on* S', *it follows, as a result of the minimizing property of harmonic functions indicated earlier, that* $U \geqq w$ *in* S. Hence, by Definition 6:1, w(x,y) *is subharmonic in* R.

It was stated in the introductory paragraph of Section 5 that for the Dirichlet integral the theorem corresponding to Theorems 5:2 and 5:3 could be proved under weaker hypotheses than those there required. The theorem to be proved is the following:

THEOREM 6:1. *Let* u(x,y) *be a function subharmonic in* R. *Then, for almost all squares* Q *in* R,

$$\int_Q u_x dy - u_y dx \geqq 0.$$

Let $C(x_0,y_0;r)$ be any circle contained with its interior $D(x_0,y_0;r)$ in N_h, and let (x,y) be the variable point $x = x_0 + \rho \cos \theta$, $y = y_0 + \rho \sin \theta$, where $0 \leqq \theta < 2\pi$, and $0 < \rho \leqq r$. Since $u^{(h)}(x,y)$ has continuous partial derivatives in N_h, we have,

$$(6:3) \quad \begin{aligned} &\frac{\partial}{\partial \rho} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \\ &= \frac{\partial}{\partial x} u^{(h)}(x,y) \cos \theta + \frac{\partial}{\partial y} u^{(h)}(x,y) \sin \theta \end{aligned}$$

in the disc $D(x_0,y_0;r)$. It now follows that for $0 < \rho \leqq r$,

$$(6:4) \quad \begin{aligned} &\frac{d}{d\rho} \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta \\ &= \frac{1}{2\pi} \int_{C(x_0,y_0;\rho)} u_x^{(h)} dy - u_y^{(h)} dx \geqq 0, \end{aligned}$$

since the integral mean

$$\frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

is a non-decreasing function of ρ [11, p. 8]. Now $u_x^{(h)}$ and $u_y^{(h)}$ are absolutely continuous in the sense of Tonelli in N_h by Property 3:8, and therefore the partial derivatives $\partial u_x^{(h)}/\partial x$, $\partial u_y^{(h)}/\partial y$ are summable in arbitrary closed subsets of N_h by Property 3:9. Under these conditions, it is permissible to apply Green's lemma to the line integral appearing in (6:4). We then obtain

$$(6:5) \quad \frac{1}{\pi \rho^2} \iint_{D(x_0,y_0;\rho)} \left(\frac{\partial}{\partial x} u_x^{(h)} + \frac{\partial}{\partial y} u_y^{(h)}\right) dxdy \geqq 0, \quad (0 < \rho \leqq r).$$

But the left-hand member of (6:5) is the integral mean of $\partial u_x^{(h)}/\partial x + \partial u_y^{(h)}/\partial y$ on $D(x_0,y_0;\rho)$. Such integral means taken over discs have essentially the same properties as those taken over square regions. In particular, Property 3:2 holds for means of this type. We may obtain an inequality similar to (6:5) for each point (x_0,y_0) in N_h. Passing to the limit in (6:5) as ρ approaches zero, we find that for every $h > 0$, the inequality

$$\frac{\partial}{\partial x} u_x^{(h)} + \frac{\partial}{\partial y} u_y^{(h)} \geqq 0 \tag{6:6}$$

holds almost everywhere in N_h. Inequality (6:6) may be written almost everywhere as

$$\frac{1}{4h^2}\int_Q u_x dy - u_y dx \geqq 0, \qquad [h > 0;\ Q = Q(x,y;h) \subset N_h]. \tag{6:7}$$

We thus see that in accordance with Definition 4:1 the inequality (6:7) holds for almost all squares in R. This completes the proof.

It has not been assumed in the above proof that the subharmonic function $u(x,y)$ be admissible; i.e., it has not been required that u_x and u_y belong to L^2. As for the converse, if u is actually continuous in $\bar{R}$, a.c.s. in R, admissible, and satisfies the Haar inequality for almost all squares of R, then by the propositions on page 41 it results that u is subharmonic in R. But we may deduce a converse of Theorem 6:1 under still weaker hypotheses. This will be done in the next theorem. We first introduce the definition of almost subharmonic function in a region. Except for the last statement, it is the same as that presented by Radó [11, p. 20].

DEFINITION 6:2. A function u is almost subharmonic in a connected region R if it satisfies the following conditions:

(i) u is summable on every measurable set contained together with its frontier in R;

(ii) *almost everywhere in* R *the inequality*

$$u(x_0,y_0) \leqq \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

holds for every r *such that* $\bar{D}(x_0,y_0;r)$ *is in* R.

If the region R *is not connected, then* u(x,y) *is almost subharmonic in* R *in case it is almost subharmonic in every connected subregion of* R.

It results from this definition that a function u is almost subharmonic in R if and only if there exists in R a subharmonic function u^* such that $u = u^*$ almost everywhere in R [11, pp. 20, 21]. Hence, we see that many of the properties herein developed for subharmonic functions may also be ascribed to almost subharmonic functions.

THEOREM 6:2. *Suppose that* u(x,y) *is summable in every open set comprised with its frontier in* R *together with its first order partial derivatives, and that in* R *the function* u *is absolutely continuous in* x *for almost every* y *and absolutely continuous in* y *for almost every* x. *If, furthermore,*

$$\int_Q u_x dy - u_y dx \geqq 0$$

for almost all squares of R, *then* u *is almost subharmonic in* R. *If, in addition,* u *is continuous in* R, i.e., *if* u *is* a.c.s. *in* R, *then it is subharmonic in* R.

The proof, for the most part, amounts to reversing the procedure in the proof of Theorem 6:1. By Property 3:7, it follows from our hypothesis that $\partial u_x^{(h)}/\partial x + \partial u_y^{(h)}/\partial y \geqq 0$ almost everywhere in N_h. By Property 3:9, the partial derivatives in the left-hand member are summable on closed subsets of N_h. If $\bar{D}(x_0,y_0;r)$ is in R, we thus have

$$\iint_{D(x_0,y_0;\rho)} \left(\frac{\partial}{\partial x} u_x^{(h)} + \frac{\partial}{\partial y} u_y^{(h)}\right) dxdy \geqq 0, \qquad (0 < \rho \leqq r).$$

By Green's lemma and Property 3:7 it follows that

$$\int_{C(x_0,y_0;\rho)} \frac{\partial}{\partial x} u^{(h)} dy - \frac{\partial}{\partial y} u^{(h)} dx \geqq 0, \qquad (0 < \rho \leqq r),$$

and the argument which led to equation (6:4) now implies

$$\frac{d}{d\rho} \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta \geqq 0, \qquad (0 < \rho \leqq r).$$

Hence the integral mean

$$\frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta$$

is a monotone non-decreasing function of ρ for each (x_0,y_0) in N_h. By the mean value theorem for integrals, and by the continuity of $u^{(h)}$, we now deduce that the inequality

$$(6{:}8) \quad u^{(h)}(x_0,y_0) \leqq \frac{1}{2\pi} \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) d\theta, \qquad (0 < \rho \leqq r),$$

holds for all (x_0,y_0) in N_h and r such that $\bar{D}(x_0,y_0;r)$ is in N_h. To show that u is almost subharmonic, integrate both sides of the above expression between the limits 0 and r. Thus,

$$\begin{aligned} u^{(h)}(x_0,y_0) &= \frac{2}{r^2} \int_0^r u^{(h)}(x_0,y_0) \rho \, d\rho \\ &\leqq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u^{(h)}(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \rho \, d\rho \, d\theta. \end{aligned}$$

And now, by Properties 3:2 and 3:5, we finally conclude that

$$u(x_0,y_0) \leqq \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} u(x_0 + \rho \cos \theta, y_0 + \rho \sin \theta) \rho \, d\rho \, d\theta$$

almost everywhere in R. Hence, $u(x,y)$ is almost subharmonic in R. If u had been taken to be continuous, then passing to the limit in (6:8) it would follow that

$$u(x_0,y_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)d\theta$$

everywhere in R; that is, u would be subharmonic in R [11, p. 7].

If $u(x,y)$ is an admissible subharmonic function, then by Theorem 6:1 and Lemma 6:1 the function u affords the Dirichlet integral an absolute minimum with respect to non-positive admissible variations. We shall now prove a stronger theorem which is analogous to the theorem of McShane referred to in an early paragraph of this section.

THEOREM 6:3. *If* $u(x,y)$ *is a subharmonic function in* R, *then* u *renders the Dirichlet integral an absolute minimum in the class of functions* $z = u + \eta$ *where* η *is continuous in* $\bar{R}$, a.c.s. *in* R, *equal to zero on* R', *and non-positive in* R.

We shall first prove the theorem for the case where $\eta \equiv 0$ in a neighborhood of the frontier R'. Indeed, let S be any open set comprised together with its frontier in R, and suppose $\eta \equiv 0$ in $\bar{R} - S$. Select any open set S_1 such that $R \supset \bar{S}_1$ and $S_1 \supset \bar{S}$. Let d be the minimum of the distances of points of S_1' from points of R' + S'. Then $z^{(h)}$ and $u^{(h)}$ are defined in $\bar{S}_1$ for $h < d/\sqrt{2}$. Also, $z^{(h)} = u^{(h)}$ on S_1' and $z^{(h)} \leqq u^{(h)}$ in S_1. Moreover, $z^{(h)}$ and $u^{(h)}$ are of class C' in $\bar{S}_1$, and $u^{(h)}$ is subharmonic in S_1. Then, by Theorem 6:1 and Lemma 6:1,

$$I[u^{(h)};S_1] \leqq I[z^{(h)};S_1]. \tag{6:9}$$

If $I[z;R] = \infty$, then the result of the theorem is trivial. Let us assume, therefore, that $I[z;R] < \infty$, so that z_x and z_y belong to L^2. Then, we easily show by Schwarz's inequality that

$$(z_x^{(h)})^2 \leqq (z_x^2)^{(h)}. \tag{6:10}$$

Now, by Fatou's lemma [14, p. 29], Properties 3:2 and 3:7, and by (6:9) and (6:10), we obtain the sequence of inequalities

$$\begin{aligned} I[u;S_1] &\leqq \liminf_{h\to 0} I[u^{(h)};S_1] \leqq \liminf_{h\to 0} I[z^{(h)};S_1] \\ &\leqq \lim_{h\to 0} \iint_{S_1} (z_x^2 + z_y^2)^{(h)}\,dxdy = I[z;S_1]. \end{aligned} \tag{6:11}$$

At each point of $R - S$ the function $z - u$ is zero, the maximum of its values in R. Hence, at every point of $R - S$ where z_x, z_y, u_x, u_y exist, we must have $z_x = u_x$, $z_y = u_y$. This occurs almost everywhere in $R - S$. Therefore, $I[u;R - S_1] = I[z;R - S_1]$ and by (6:11),

$$I[u;R] \leqq I[z;R]. \tag{6:12}$$

If we had $I[u;R] = \infty$, then clearly we must have $I[z;R] = \infty$; otherwise the sequence of inequalities in (6:11) would lead to a contradiction in (6:12).

The remainder of the argument consists in extending the result just obtained to any function $z = u + \eta$ satisfying the conditions of the theorem. For harmonic functions McShane [6, p. 595] performs this extension in a very efficient and concise manner, so the particulars will not be repeated here.

Recapitulating some of the results of this section, we see that there is an association between subharmonic functions and the calculus of variations. For *continuous* subharmonic functions a number of characterizing properties may be stated which have interest from the point of view of the calculus of variations. *A function* $u(x,y)$ *which is continuous in* R *has been found to be subharmonic in* R *if and only if it satisfies any one of the following equivalent conditions:*

(i) *it fulfills condition* (iv) *of Definition* 6:1;

(ii) *it is* a.c.s. *in* R, *its first order partial derivatives are summable in every closed subset of* R, *and* $\int_Q u_x dy - u_y dx \geqq 0$ *for almost all squares of* R;

(iii) *it is* a.c.s. *in* R *and renders the Dirichlet integral a proper minimum in the class of functions* $z = u + \eta$, *where* η *is continuous in* $\bar{R}$, a.c.s. *in* R, *equal to zero on* R', *and non-positive in* R.

Actually, the necessity of the last property was proved in Theorem 6:3 under somewhat weaker assumptions on u and z than those stated here.

7. *Association with the problem of Plateau in non-parametric form.* The theory of Section 5 may be applied without difficulty to the integrand $f(p,q) = (1 + p^2 + q^2)^{1/2}$. It is clear that this problem is positive regular, for $f_{pp} = (1+q^2)/f^3$, $f_{qq} = (1+p^2)/f^3$, and $f_{pp}f_{qq} - f_{pq}{}^2 = 1/f^2$.

Let R be a Jordan region; in particular, for simplicity we shall take R as a rectangle R: $a < x < b$; $c < y < d$. If $z = Z(x,y)$ is continuous in $\bar{R}$ and has a finite area in the sense of Lebesgue [9, p. 6], then this area is given by the integral

$$I = \iint_R (1 + z_x{}^2 + z_y{}^2)^{1/2} dxdy$$

if and only if z is A.C.T. [9, p. 10][*]. If Z(x,y) renders the area integral a minimum in the class of all Lipschitzian functions defined in $\bar{R}$ and assuming the same boundary values, then Z(x,y) is analytic and, for these boundary values, is a solution of the Problem of Plateau; i.e., Z(x,y) satisfies the partial differential equation

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[*] For the case of the general Jordan region, see E. J. McShane, *Integrals over surfaces in parametric form.* Annals of Mathematics (2) vol. 34 (1933), pp. 815-838; in particular, p. 836.

$$(7{:}1) \qquad (1 + z_x^2)z_{yy} - 2z_xz_yz_{xy} + (1 + z_y^2)z_{xx} = 0$$

in R [9, p. 61]. Except for a multiplicative factor $1/f^3$, equation (7:1) is the Euler equation for this calculus of variations problem and is equivalent to the statement that the mean curvature of Z is everywhere zero. The surface Z(x,y) is then a minimal surface.

In general, if S is the interior of a convex Jordan curve S', then there exists in S a minimal surface of the form $z = Z_0(x,y)$ taking on prescribed continuous boundary values on S', and $z = Z_0(x,y)$ is a minimal surface whose area is actually a minimum with respect to all continuous surfaces of the type of the circular disc, having the same boundary values on S' [9, p. 101]. This fact is used to obtain the concluding inequality (7:2) of this section.

In view of the nature of the integrand function f(p,q) it is seen that Z is admissible (∞) if it is single valued and measurable in $\bar{R}$, and if its first order partial derivatives are summable in R. The corresponding admissible variations are functions $\eta(x,y)$, continuous in $\bar{R}$, a.c.s. in R, vanishing on R', and whose first order partial derivatives are summable in R. In particular, it is to be noted that admissible variations are A.C.T. in R.

We might observe that the first variation reduces for the case here treated to

$$I_1[\eta] = \iint_R \frac{1}{f(Z_x,Z_y)} (\eta_x Z_x + \eta_y Z_y)dxdy,$$

and the Haar inequality becomes

$$\int_Q \frac{1}{f(Z_x,Z_y)} (Z_x dy - Z_y dx) \geqq 0.$$

Plainly, the results of earlier sections may be stated so as to be applicable in the present discussion. Corresponding to

Theorem 4:1 we conclude that $I_1[\eta] \geq 0$ *for non-positive admissible variations if and only if the Haar inequality holds*. Moreover, it follows at once from Theorem 5:2 that if $I_1[\eta] \geq 0$ *for non-positive admissible variations, or if the Haar inequality holds in* R, *then* Z *minimizes the integral* I *with respect to non-positive admissible variations*.

As a consequence of Theorem 5:2 and the opening part of this section we note that if *an admissible* Z *is continuous in* $\bar{R}$, *of class* $C^{(2)}$ *in* R, *and if in every disc* D(x,y;r) *in* R, *we have* $Z \leq W$, *where* W(x,y) *is the minimal surface coinciding with* Z *on* C(x,y;r), *then* $I_1[\eta] \geq 0$ *for all non-positive admissible variations*.

It is now evident that if the surfaces under consideration be continuous in $\bar{R}$ and of class $C^{(2)}$ in R, then the hypotheses and conclusions appearing in the underlined propositions of this section are equivalent. By the method employed in the proof of Theorem 5:2 we may then derive an inequality analogous to the Euler equation (7:1), namely,

$$(1 + Z_x^2)Z_{yy} - 2Z_xZ_yZ_{xy} + (1 + Z_y^2)Z_{xx} \geq 0, \tag{7:2}$$

which must hold at all points of R. This implies that the mean curvature of Z is non-negative throughout R.

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DIFFERENTIAL SYSTEMS INVOLVING k-POINT BOUNDARY CONDITIONS

BY

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DIFFERENTIAL SYSTEMS INVOLVING k-POINT BOUNDARY CONDITIONS

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1. Introduction. This paper is concerned with a k-point boundary value problem consisting of a system of ordinary linear differential equations and a set of boundary conditions involving linearly the values of the solutions at interior points as well as at the end points of the interval over which the system is defined.

By means of a transformation previously employed by Reid [6]* and Denbow [3] this boundary value problem is reduced, in Section 2, to a two-point boundary problem. Known results for this reduced problem imply analogous results for the original k-point system. By this device, the fundamental properties of the k-point boundary value problem and its adjoint system are readily deduced. The definition of self-adjoint, definitely self-adjoint, self-conjugate adjoint, and definitely self-conjugate adjoint systems as presented by Bliss ([1] and [2]) and Reid [10] are extended in Section 3 to the k-point boundary value problem under consideration.

Section 4 deals with necessary and sufficient conditions for such k-point boundary value problems to arise from the calculus of variations. Hu [4] has considered this question for the two-point case and arrived at certain necessary and sufficient conditions. These conditions of Hu are now extended to the k-point case.

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*Numbers in square brackets refer to the bibliography at the end of this paper.

Some relationships between the work of this paper and a recent paper by Langer [5], dealing with boundary value problems in the complex domain, are pointed out in Section 5. The manner in which the adjoint system is obtained by the method of this paper leads to the same relationships for the adjoint system and the form of the Green's matrix as established by Langer. The methods employed here are thus shown to be capable of achieving results in the complex domain.

In passing, it is to be mentioned that the present work is related to an earlier investigation of k-point boundary value problems by Wilder [14]. Toyoda ([12] and [13]) also considered the Green's function in space of one dimension. Toyoda was apparently unaware of Wilder's previous work since, although his treatment is strikingly similar to that of Wilder, no reference was made to the latter's paper. In the discussions of both Wilder and Toyoda the amount of computation seemed to be unnecessarily complicated. The work of this paper leads to desirable simplification in the work of these writers.

2. Differential systems involving k-point boundary conditions. Suppose we have given a system of n first order linear differential equations, together with a system of $\nu = (k-1)n$ boundary conditions applying at k points

$$a = a_1 < a_2 < \dots < a_k = b,$$

of the real interval ab: $a \leq x \leq b$. The differential equations and boundary conditions are written in the form

$$y_i' = \sum_{j=1}^{n} A_{ij}(x) y_j,$$

$$(2{:}1)\quad s_\sigma[y] \equiv \sum_{q=1}^{k-1} \sum_{j=1}^{n} \left[M_{\sigma j:q} y_j(a_q^+) + N_{\sigma j:q} y_j(a_{q+1}^-) \right] = 0,$$

$$(i = 1,2,\dots,n;\ \sigma = 1,2,\dots,\ \nu = (k-1)n).$$

The functions $A_{ij}(x)$ are supposed to be continuous on the interval ab, except possibly at the intermediate points $a_2, a_3, \ldots, a_{k-1}$; throughout this paper, however, it is assumed that these functions approach finite right- and left-hand limits at these intermediate points. In the boundary conditions the notations $y_i(a_q^+)$, $y_i(a_q^-)$ are abbreviations for the right- and left-hand limits, respectively, of $y_i(x)$ at the point $x = a_q$. Since we shall be interested in the solutions of (2:1) only on the interval ab, we shall often write $y_i(a)$, $y_i(b)$ in place of $y_i(a^+)$, $y_i(b^-)$, respectively.

By a solution of (2:1) we shall understand a set of functions $y_i(x)$ which are continuous and have continuous derivatives on each open sub-interval $a_q < x < a_{q+1}$, $(q = 1,2,\ldots,k-1)$, which satisfy the differential equations of (2:1) on each of these sub-intervals, and whose values $y_i(a_q^+)$, $y_i(a_{q+1}^-)$ satisfy the boundary conditions of (2:1). These right- and left-hand limits clearly exist in view of the above hypothesis on the behavior of $A_{ij}(x)$ at the intermediate points $a_2, a_3, \ldots, a_{k-1}$. It is to be emphasized that in general a solution of (2:1) is not continuous across the intermediate points $a_2, a_3, \ldots, a_{k-1}$; that is, in general $y_i(a_q^+) \neq y_i(a_q^-)$, $(q = 2,3,\ldots,k-1)$. In particular, a solution of this system is non-trivial if there is at least one sub-interval $a_q < x < a_{q+1}$ on which not all the components $y_i(x)$ are identically zero. Similarly, two solutions of this system are linearly dependent if and only if there is a non-trivial linear combination of them which is identically zero on the interval ab. It is to be understood throughout that the boundary conditions $s_\sigma[y]$ constitute a set of ν independent linear forms in the 2ν values $y_i(a)$, $y_i(a_2^-)$, $\ldots$, $y_i(a_{k-1}^+)$, $y_i(b)$.

For simplicity, the subsequent discussion will be largely presented in matrix notation. Roman capitals will be used to denote n x n matrices; $\nu \times \nu$ matrices will be represented by script capitals. The subscripts i, j will have the range 1, 2, ..., n, while the subscripts σ, τ will have the range 1, 2,..., ν. Also, i and σ will be used for row subscripts; j and τ for column subscripts. The lower case letters y, z will be used for vectors of n components while u, v will be used for vectors of ν components. The vectors $(M_{ij}y_j)$ and (y_jM_{ji}), where the repetition of the subscript denotes summation with respect to this index over its range of definition, are denoted by My and yM, respectively. Similarly, $\mathcal{M}u$ and $u\mathcal{M}$ denote the vectors $(\mathcal{M}_{\sigma\tau}u_\tau)$ and $(u_\tau \mathcal{M}_{\tau\sigma})$, respectively. The letter s will be used to denote a vector of ν components. In case some other letter is used as a vector, the number of components will be specified. Whenever occasion arises, use will be made of the special symbols $M_{:q} \equiv \|M_{\sigma j:q}\|$, $N_{:q} \equiv \|N_{\sigma j:q}\|$, to denote rectangular matrices of ν rows and n columns. The symbols $P_{q:} \equiv \|P_{q:i\tau}\|$, $Q_{q:} \equiv \|Q_{q:i\tau}\|$ will denote rectangular matrices of n rows and ν columns. The conjugate, the transpose, and the conjugate transpose of a matrix M will be denoted by $\bar{M}$, $\breve{M}$ and M^*, respectively. The index q will be understood to have the range 1, 2, ..., k - 1 unless the range is otherwise indicated. Hereinafter subscripts will be omitted while making use of the foregoing notations, except where it may be more advisable for the subject under discussion to make use of the subscripts with indication of their ranges. We may now write system (2:1) as

$$\begin{aligned} & y' = A(x)y, \\ (2:2) \qquad & s[y] \equiv \sum_{q=1}^{k-1}[M_{:q}y(a_q^+) + N_{:q}y(a_{q+1}^-)] = 0. \end{aligned}$$

A change of independent variable is now introduced into equations (2:2) in order to transform each of the k - 1 sub-intervals

$a_q a_{q+1}$ individually into the interval $0 \leqq t \leqq 1$. This may be accomplished by the transformation $x = x_q[t]$, $0 \leqq t \leqq 1$, where

$$(2:3) \qquad x_q[t] = a_q + t(a_{q+1} - a_q), \quad 0 \leqq t \leqq 1.$$

With (2:3), let the set of functions $u_\sigma(t)$ be defined by the equations

$$(2:4) \quad u_{(q-1)n+i}(t) = y_i(x_q[t]), \quad (q = 1,2,\dots,k-1;\ i = 1,2,\dots,n).$$

In particular, we may write

$$(2:5) \qquad \begin{aligned} y_i(a_q^+) &= u_{(q-1)n+i}(0), \\ y_i(a_{q+1}^-) &= u_{(q-1)n+i}(1). \end{aligned}$$

We shall now consider the differential system with two-point boundary conditions satisfied by a vector $u(t) \equiv (u_\sigma(t))$ defined by (2:4), (2:5) in terms of a solution $y(x)$ of (2:2). Writing u'_σ and x'_q for the derivatives of u_σ and x_q with respect to the variable t defined by (2:3), we have

$$(2:6) \quad u'_{(q-1)n+i} = \frac{d}{dt}\, y_i(x_q[t]) = y'_i(x_q[t])x'_q[t] = y'_i(x_q[t])\cdot(a_{q+1} - a_q).$$

Now define the matrix $\mathcal{A}(t)$ as

$$(2:7) \quad \mathcal{A}(t) \equiv \left\| \begin{matrix} (a_2 - a_1)A(x_1[t]) & 0 & \cdots & 0 \\ 0 & (a_3 - a_2)A(x_2[t]) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (a_k - a_{k-1})A(x_{k-1}[t]) \end{matrix} \right.$$

If we set $\mathcal{A}_{\sigma\tau}(0) = \mathcal{A}_{\sigma\tau}(0^+)$ and $\mathcal{A}_{\sigma\tau}(1) = \mathcal{A}_{\sigma\tau}(1^-)$, then clearly the elements of this matrix are continuous on $0 \leqq t \leqq 1$. Correspondingly, if $\mathcal{M}$ and $\mathcal{N}$ are defined as

$$\begin{aligned} \mathcal{M} &\equiv \| M_{:1}, M_{:2}, \ldots, M_{:k-1} \|, \\ \mathcal{N} &\equiv \| N_{:1}, N_{:2}, \ldots, N_{:k-1} \|, \end{aligned} \tag{2:8}$$

the independence of the linear forms $s_\sigma [y]$ is seen to imply that the $\nu \times 2\nu$ matrix $\| \mathcal{M}, \mathcal{N} \|$ is of rank ν. With the matrices $\mathcal{A}(t)$, $\mathcal{M}$ and $\mathcal{N}$ thus defined, the vector $u(t)$ defined by (2:4) satisfies the vector differential system

$$\begin{aligned} u' &= \mathcal{A}(t) u, \\ s[u] &\equiv \mathcal{M} u(0) + \mathcal{N} u(1) = 0. \end{aligned} \tag{2:9}$$

This latter system, involving two-point boundary conditions, is of the form considered by Bliss [1].

Conversely, if $u \equiv (u_\sigma(t))$ is a solution of (2:9), then the related $y(x) \equiv (y_i(x))$ determined by (2:4) is readily seen to be a solution of (2:1).

A system adjoint to (2:9) is (Bliss [1], p. 565), by definition,

$$\begin{aligned} v' &= -v \mathcal{A}(t), \\ r[v] &\equiv v(0) \mathcal{P} + v(1) \mathcal{Q} = 0, \end{aligned} \tag{2:10}$$

where the matrices $\mathcal{P}$, $\mathcal{Q}$ occurring in the boundary conditions $r[v] = 0$ are selected so that

$$\mathcal{M} \mathcal{P} - \mathcal{N} \mathcal{Q} = 0 \tag{2:11}$$

is satisfied, and the $2\nu \times \nu$ matrix

$$\left\| \begin{matrix} \mathcal{P} \\ \mathcal{Q} \end{matrix} \right\|$$

is of rank ν. It may be readily verified that the adjoint system (2:10) is unique except for a non-singular linear transformation on the boundary conditions. It is advantageous to also write

$$(2{:}12) \qquad \mathcal{P} \equiv \left\| \begin{matrix} P_{1:} \\ P_{2:} \\ \cdots \\ \cdots \\ \cdots \\ P_{k-1:} \end{matrix} \right\| \qquad \mathcal{Q} \equiv \left\| \begin{matrix} Q_{1:} \\ Q_{2:} \\ \cdots \\ \cdots \\ \cdots \\ Q_{k-1:} \end{matrix} \right\|$$

where $P_{q:}$ and $Q_{q:}$ are $n \times \nu$ matrices as indicated above. The condition (2:11) is readily seen to be equivalent to

$$(2{:}13) \qquad \sum_{q=1}^{k-1} [M_{:q}P_{q:} - N_{:q}Q_{q:}] = 0.$$

If $\|Y_{ij:q}(x)\|$ is a fundamental matrix solution of the differential equations of (2:2) on the interval $a_q a_{q+1}$ and $Y^{(q)}(t) \equiv \|Y_{ij:q}(x_q[t])\|$, then

$$(2{:}14) \qquad \mathcal{U}(t) \equiv \left\| \begin{matrix} Y^{(1)}(t) & 0 & \cdots & 0 \\ 0 & Y^{(2)}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Y^{(k-1)}(t) \end{matrix} \right\|$$

is seen to be a fundamental matrix solution of the differential equations of (2:9) on $0 \leqq t \leqq 1$. That is, $u \equiv \mathcal{U}_\tau \equiv (\mathcal{U}_{\sigma\tau})$, $(\tau = 1, 2, \ldots, \nu)$, are ν linearly independent solutions of this vector differential equation. The most general solution of the vector differential equation of (2:9) is then of the form $u(t) = \mathcal{U}(t)c$, where $c \equiv (c_\sigma)$ is an arbitrary constant vector. Moreover, a vector $u_\sigma = \mathcal{U}_{\sigma\tau}(t)c_\tau$ is a solution of (2:10) if and only if

$$y_i(x) = \sum_{j=1}^{n} Y_{ij:q}(x)c_{(q-1)n+j}, \qquad (a_q \leqq x \leqq a_{q+1})$$

is a solution of (2:2). Now define the matrix $\mathcal{D}$ as

$$(2{:}15) \qquad \mathcal{D} \equiv \| s_\sigma[\mathcal{U}_\tau] \| \equiv \| \mathcal{M}\,\mathcal{U}(0) + \mathcal{N}\,\mathcal{U}(1) \|.$$

In view of the above definition of $\mathcal{U}(t)$, one may also write

$$(2{:}16) \qquad \mathcal{D} \equiv \| D_{:1}, D_{:2}, \dots, D_{:k-1} \|,$$

where

$$D_{:q} \equiv M_{:q}Y_{:q}(a_q) + N_{:q}Y_{:q}(a_{q+1}),$$

it being understood in this last formula that there is no summation with respect to q. It is at once apparent that compatibility conditions for the k-point system, as well as for the two-point system, may be stated in the terms of the rank of $\mathcal{D}$.

The system (2:10) may be transformed to obtain the system adjoint to (2:2) on the interval ab. Define $t_q[x]$ as the inverse of $x_q[t]$ and let

$$(2{:}17) \quad z_i(x) = v_{(q-1)n+i}(t_q[x]), \quad (a_q < x < a_{q+1}; q = 1,2,\dots,k-1)$$

Clearly the functions $z_i(x)$ are continuous and have continuous derivatives on ab except possibly at the intermediate values a_2, a_3, ..., a_{k-1}. At these points, however, these functions assuredly have finite right- and left-hand limits. It is then readily computed that the vector z(x) satisfies the differential system

$$z' = -zA(x),$$

$$(2{:}18) \qquad r[z] \equiv \sum_{q=1}^{k-1} [z(a_q^+)P_{q:} + z(a_{q+1}^-)Q_{q:}] = 0,$$

where $P_{q:}$ and $Q_{q:}$ are related to the matrices $M_{:q}$ and $N_{:q}$ of (2:2) by equation (2:13).

Consider now the non-homogeneous system

$$(2{:}19)\qquad \begin{aligned} y' &= A(x)y + f(x), \\ s[y] &= h, \end{aligned}$$

where the components of the vector $f(x) \equiv (f_i(x))$ are continuous on ab except possibly at the intermediate points $x = a_2, a_3, \ldots, a_{k-1}$, and $h \equiv (h_\sigma)$ is a given set of constants. When this problem is transformed by (2:3), (2:4), we have the corresponding non-homogeneous vector system

$$(2{:}20)\qquad \begin{aligned} u' &= \mathcal{A}(t)u + g(t), \qquad (0 \leqq t \leqq 1), \\ s[u] &= h, \end{aligned}$$

where the vector $g(t) \equiv (g_\sigma(t))$ is defined by

$$(2{:}21)\ g_{(q-1)n+i}(t) = (a_{q+1} - a_q)f_i(x_q[t]), (q = 1, 2, \ldots, k-1; i = 1, 2, \ldots, n$$

The situation is analogous to that of the homogeneous case in that a vector solution of (2:20) implies a corresponding vector solution of (2:19).

With the preceding remarks and definitions, we now consider the results that have been proved and discussed by Bliss [1] and others for the two-point case. In particular, Bliss has shown that if $\mathcal{B}$ is non-singular, the only solution of system (2:9) is $u(t) \equiv 0$, and for the non-homogeneous problem (2:20) there exists one and only one solution. If $\mathcal{B}$ has rank $\nu - \rho$, then (2:9) has exactly ρ linearly independent solutions, and the number of such solutions for (2:9) is always the same as the number for the adjoint system (2:10). Also, if $\mathcal{B}$ has rank $\nu - \rho$ the non-homogeneous system (2:20) has solutions if and only if the equation

$$\int_0^1 v(t)g(t)dt = r^*[v]h \tag{2:22}$$

is satisfied for every solution $v(t)$ of the adjoint system (2:10). In (2:22), $r^*[v] \equiv (r^*_\sigma [v])$ is a certain vector whose components are linear forms in the end values of $v(t) \equiv (v_\sigma(t))$ at $t = 0$ and $t = 1$. Bliss ([1], p. 567) has established this result for the case $h = 0$. However, the result in general is proved by quite similar argument. In (2:22) the vector $r^*[v]$ corresponds to the vector $\bar{T}(z) \equiv (\bar{T}_1(z))$ defined in Section 1 of Bliss [1].

From these known results we may now deduce for the k-point system analogous conclusions, which will be stated in the form of theorems.

THEOREM 2:1. *If the matrix* $\mathcal{D}$ *is non-singular, the only solution of the boundary value problem* (2:2) *is* $y(x) \equiv 0$, $a \leqq x \leqq b$. *If* $\mathcal{D}$ *has rank* $\nu - \rho$ *then the problem has exactly* ρ *linearly independent solutions on this interval; in this case the adjoint problem* (2:18) *also has exactly* ρ *linearly independent solutions*.

The proof of the following theorem is deduced from Bliss ([1], p. 567).

THEOREM 2:2. *If the matrix* $\mathcal{D}$ *is non-singular, a corresponding non-homogeneous system* (2:19) *has one and only one solution on* $a \leqq x \leqq b$.

If $y(x)$ is a solution of (2:19), it follows from (2:17) that condition (2:22) for the corresponding system (2:20) is equivalent to the condition that

$$\int_a^b z(x)f(x)dx \equiv \sum_{q=1}^{k-1} \int_{a_q}^{a_{q+1}} z(x)f(x)dx = r^*[z]h$$

hold for every solution $z(x)$ of the adjoint system (2:18). Consequently, we have the following result for the k-point problem.

THEOREM 2:3. *If the matrix* $\mathcal{D}$ *has rank* $\nu - \rho$, *then the system* (2:19) *has solutions if and only if the equation*

$$\int_a^b z(x)f(x)dx = r^*[z]\,h \tag{2:23}$$

is satisfied for every solution $z(x)$ *of the adjoint system* (2:18). *The most general solution of* (2:19) *is then*

$$y(x) = y^{(0)}(x) + y^{(1)}(x)c_1 + \dots + y^{(\rho)}c_\rho, \tag{2:24}$$

where the $y^{(p)}(x)$, $(p = 1,2,\dots,\rho)$, *are* ρ *linearly independent solutions of* (2:2).

Now consider the two-point system (2:9) with the determinant of $\mathcal{D}$ supposed different from zero. The existence and uniqueness of the Green's matrix for such an incompatible system has been proved by several writers. The Green's matrix is a matrix $\mathcal{G}(t,\xi) \equiv \|\mathcal{G}_{\sigma\tau}(t,\xi)\|$ whose elements are single-valued for $0 \leqq t \leqq 1$, $0 \leqq \xi \leqq 1$. The elements are, furthermore, continuous and have continuous derivatives in t except along the line $t = \xi$. This matrix has the following well-known properties:

$$\mathcal{G}(\xi^+,\xi) - \mathcal{G}(\xi^-,\xi) = \mathcal{I}, \tag{2:25}$$

$$\frac{\partial}{\partial t}\mathcal{G}(t,\xi) = \mathcal{A}(t)\mathcal{G}(t,\xi), \quad 0 \leqq t \leqq \xi, \quad \xi \leqq t \leqq 1, \tag{2:26}$$

$$\mathcal{M}\mathcal{G}(0,\xi) + \mathcal{N}\mathcal{G}(1,\xi) = 0. \tag{2:27}$$

In (2:25), $\mathcal{I}$ denotes the $\nu \times \nu$ identity matrix.

The properties (2:26), (2:27) show that the columns of the matrix $\mathcal{G}(t,\xi)$ would be solutions of (2:9) of the kind demanded by the boundary conditions of this system if it were not for their discontinuities at $t = \xi$. If we put

(2:28) $$\Delta \equiv \mathcal{M}\mathcal{U}(0) - \mathcal{N}\mathcal{U}(1),$$

the explicit form of the Green's matrix for an incompatible system (2:9) is

(2:29) $$\mathcal{G}(t,\xi) = \frac{1}{2}\,\mathcal{U}(t)\left[\frac{|t-\xi|}{t-\xi}\mathcal{I} + \mathcal{D}^{-1}\Delta\right]\mathcal{U}^{-1}(\xi).$$

The corresponding matrix $\mathcal{H}(t,\xi) = -\breve{\mathcal{G}}(\xi,t)$ may be shown to be the Green's matrix of the adjoint system (2:10). As a function of its second argument, the Green's matrix has the following properties:

(2:30) $$\mathcal{G}(t,t^-) - \mathcal{G}(t,t^+) = \mathcal{I},$$

(2:31) $$\frac{\partial}{\partial\xi}\mathcal{G}(t,\xi) = -\mathcal{G}(t,\xi)\mathcal{A}(\xi), \qquad 0 \leqq \xi < t,\ t < \xi \leqq 1,$$

(2:32) $$\mathcal{G}(t,0)\mathcal{P} + \mathcal{G}(t,1)\mathcal{Q} = 0.$$

Moreover, if the components of the vector $g(t)$ are continuous on the interval $0 \leqq t \leqq 1$, then the unique solution $u(t)$ of the non-homogeneous system (2:20) with $h = (0_\sigma)$ is given by (see Bliss [1], Section 5)

(2:33) $$u(t) = \int_0^1 \mathcal{G}(t,\xi)g(\xi)d\xi.$$

Now for a solution $u(t)$ of (2:20) it follows from (2:33) that

$$u_\sigma(t) = \sum_{p=1}^{k-1}\sum_{j=1}^{n}\int_0^1 \mathcal{G}_{\sigma,(p-1)n+j}(t,\xi)f_j(x_p[\xi])x_p'[\xi]\,d\xi$$

(2:34) $$= \sum_{p=1}^{k-1}\sum_{j=1}^{n}\int_{a_p}^{a_{p+1}} \mathcal{G}_{\sigma,(p-1)n+j}(t,\xi_p[s])f_j(s)ds.$$

Hence, on $a_q < x < a_{q+1}$,

$$u_{(q-1)n+i}(t_q[x])$$

(2:35) $$= \sum_{p=1}^{k-1}\sum_{j=1}^{n}\int_{a_p}^{a_{p+1}} \mathcal{G}_{(q-1)n+i,(p-1)n+j}(t_q[x], \xi_p[s])f_j(s)ds.$$

But, as pointed out above, a vector $u(t)$ is a solution of (2:20) if and only if the associated vector $y(x) \equiv (y_i(x))$, where $y_i(x) = u_{(q-1)n+i}(t_q[x])$, $a_q \leqq x \leqq a_{q+1}$, is a solution of (2:19). That is, the matrix $G(x,s)$ defined on $a \leqq x \leqq b$, $a \leqq s \leqq b$, except along the lines $x = a_q$, $t = a_p$, $(q,p = 2,3,\dots,k-1)$, by the equation

(2:36) $$G(x,s) \equiv \| G_{ij}(x,s)\| \equiv \|\mathcal{G}_{(q-1)n+i,(p-1)n+j}(t_q[x], \xi_p[s])\|,$$
$$(a_q < x < a_{q+1};\ a_p < s < a_{p+1};\ q,p = 1,2,\dots, k-1),$$

satisfies the following resolvent condition.

THEOREM 2:4. If the homogeneous system (2:2) is incompatible, then the unique solution of an associated non-homogeneous system (2:19) is given by

(2:37) $$y(x) = \int_a^b G(x,s)f(s)ds, \qquad a \leqq x \leqq b.$$

The matrix $G(x,s)$ defined by (2:36) is seen to have properties analogous to (2:30), (2:31), (2:32) on $a \leqq x \leqq b$, $a \leqq s \leqq b$. The corresponding matrix $H(x,s) = -\breve{G}(s,x)$ may be verified to be the Green's matrix of the adjoint system (2:18). As a function of its second argument, the Green's matrix has the following properties:

(2:38) $$G(x,x^+) - G(x,x^-) = I,$$

(2:39) $$\frac{\partial}{\partial s} G(x,s) = -G(x,s)A(s),$$

(2:40) $$\sum_{q=1}^{k-1} [G(x,a_q^+)P_{q:} + G(x,a_{q+1}^-)Q_{q:}] = 0.$$

We shall now consider the particular problem of finding solutions of the differential equations of (2:2) which are continuous on the entire interval ab: $a = a_1 \leqq x \leqq a_k = b$ and which satisfy a set of n linearly independent end conditions

$$\sum_{q=1}^{k} \sum_{j=1}^{n} m_{ij:q} y_j(a_q) = 0, \qquad (i = 1,2,\dots,n). \tag{2:41}$$

By identifying $y_j(a_q^-) = y_j(a_q^+) = y_j(a_q)$, we may write these boundary conditions and the condition of continuity at the intermediate points as

$$\begin{gathered} s_i[y] \equiv \sum_{j=1}^{n} m_{ij:1} y_j(a_1^+) + \sum_{j=1}^{n} \sum_{q=2}^{k} m_{ij:q} y_j(a_q^-) = 0, \\ s_{(p-1)n+i}[y] \equiv y_i(a_p^-) - y_i(a_p^+) = 0, \\ (i = 1,2,\dots,n;\ p = 2,3,\dots,k-1). \end{gathered} \tag{2:42}$$

When these boundary conditions are transformed to the two-point form (2:9), we have the following expressions for $\mathcal{M}$ and $\mathcal{N}$,

$$\mathcal{M} \equiv \left\| \begin{matrix} m_{ij:1} & 0 & \dots & 0 \\ 0 & -I & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I \end{matrix} \right\|,$$

(2:43)

$$\mathcal{N} \equiv \left\| \begin{matrix} m_{ij:2} & m_{ij:3} & \dots & m_{ij:k-1} & m_{ij:k} \\ I & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 \end{matrix} \right\|$$

In general, the solutions of the corresponding adjoint system (2:18) are not continuous across the intermediate values $a_2, a_3, \dots, a_{k-1}$, as will be seen from the determination of the adjoint boundary conditions. Now for a given set of boundary conditions (2:1) the corresponding adjoint boundary conditions (2:18) have their coefficients determined by (2:13). For this particular system relation (2:13) becomes

$$
\begin{aligned}
&\sum_{j=1}^{n} m_{ij:1}P_{1:j\tau} - \sum_{j=1}^{n}\sum_{q=2}^{k} m_{ij:q}Q_{(q-1):j\tau} = 0,\\
(2{:}44)\qquad &P_{(q+1):i\tau} + Q_{q:i\tau} = 0,\\
&(q = 1,2,\dots, k-2;\ i = 1,2,\dots,n;\ \tau = 1,2,\dots,\nu).
\end{aligned}
$$

In general, a set $z_i(a_q^+)$, $z_i(a_{q+1}^-)$, $(q = 1,2,\dots, k-1)$, satisfies the adjoint boundary conditions $r[z] = 0$ if and only if there exist constants (d_σ), $(\sigma = 1,2,\dots,\nu)$, such that

$$z_i(a_q^+) \equiv d_\sigma M_{\sigma j:q}, \qquad z_i(a_{q+1}^-) \equiv d_\sigma N_{\sigma j:q}.$$

For the particular system given above, a vector $z(x)$ satisfies the adjoint boundary conditions if and only if there exist constants $(d_\sigma) \equiv (d_i, d_{n+i}, \dots, d_{(k-1)n+i})$ such that

$$
(2{:}45)\qquad
\begin{aligned}
z_j(a_1^+) &= d_i m_{ij:1},\\
z_j(a_k^-) &= -d_i m_{ij:k},\\
z_j(a_q^+) &= -d_{(q-1)n+j},\\
z_j(a_q^-) &= -d_{(q-1)n+j} - d_i m_{ij:q}, \quad (q = 2,3,\dots,k-1).
\end{aligned}
$$

Equations (2:45) are seen to be equivalent to the existence of constants (d_i) such that

$$
(2{:}46)\qquad
\begin{aligned}
z_j(a_1^+) &= d_i m_{ij:1},\\
z_j(a_k^-) &= -d_i m_{ij:k},\\
z_j(a_q^-) - z_j(a_q^+) &= -d_i m_{ij:q}, \quad (q = 2,3,\dots,k-1).
\end{aligned}
$$

Corresponding to a vector $z(x)$, $a \leqq x \leqq b$, which satisfies $r[z] = 0$, define k solutions $z^{(1)}(x)$, $z^{(2)}(x)$, ..., $z^{(k)}(x)$ of

$$(2{:}47)\qquad z'(x) = -z(x)A(x), \qquad a \leqq x \leqq b,$$

whose elements are continuous on ab, and satisfy corresponding initial conditions,

$$(2{:}48)\qquad \begin{aligned} z^{(1)}(a_1) &= z(a_1{}^+), \\ z^{(k)}(a_k) &= -z(a_k{}^-), \\ z^{(q)}(a_q) &= z(a_q{}^+) - z(a_q{}^-), \quad (q = 2,3,\ldots,k-1). \end{aligned}$$

We then have that these k continuous solutions satisfy the conditions

$$(2{:}49)\qquad z_j{}^{(q)}(a_q) = d_i m_{ij:q}, \qquad (q = 1,2,\ldots,k).$$

On each sub-interval $a_p < x < a_{p+1}$,

$$(2{:}50)\qquad z(x) = z^{(1)}(x) + z^{(2)}(x) +\ldots+ z^{(p)}(x), \quad (p = 1,2,\ldots,k-1).$$

In particular,

$$z(x) = z^{(1)}(x) + z^{(2)}(x) + \ldots + z^{(k-1)}(x), \quad a_{k-1} < x < a_k,$$

and since $z^{(k)}(a_k) = -z(a_k{}^-)$, we have that

$$(2{:}51)\qquad \sum_{q=1}^{k} z^{(q)}(a_k) = 0.$$

Since each of the solutions $z^{(1)}(x)$, $z^{(2)}(x), \ldots, z^{(k)}(x)$ is continuous on ab, we have

$$(2{:}52)\qquad \sum_{q=1}^{k} z^{(q)}(x) \equiv 0,$$

on ab.

Conversely, suppose there exist k continuous solutions of (2:47) on ab which satisfy equation (2:52), and for which there exist associated constants (d_i) satisfying relations (2:49). Then z(x) defined on each sub-interval $a_p < x < a_{p+1}$ by (2:50) is seen to satisfy with these continuous solutions the point conditions (2:48), and hence also conditions (2:46), But this latter condition, as pointed out above, is equivalent to the condition that $z(x)$ satisfy the adjoint boundary conditions. We have proved, therefore, the following result.

THEOREM 2:5. *For the boundary problem consisting of the differential equations* (2:2) *and the boundary conditions* (2:41), *with the further restriction that each solution function be continuous across the intermediate values* $a_2, a_3, \ldots, a_{k-1}$, *the adjoint boundary problem is equivalent to the existence of* k *solutions* $z^{(1)}(x), z^{(2)}(x), \ldots, z^{(k)}(x)$ *of the adjoint equation* (2:47) *which are continuous throughout* $a \leqq x \leqq b$, *and which satisfy the point conditions* (2:49) *and the identity* (2:52) *on* ab.

If the boundary conditions of the adjoint problem, as algebraic equations, imply that every vector $z(x)$, $a \leqq x \leqq b$, satisfying them must be continuous across the intermediate points $a_2, a_3, \ldots, a_{k-1}$, that is, $z(a_q-) = z(a_q+)$, $(q = 2,3,\ldots, k-1)$, then it follows from (2:46) that $m_{ij:q} \equiv 0$, $(i,j = 1,2,\ldots,n;$ $q = 2,3,\ldots, k-1)$. That is, the original system must reduce to one with two-point boundary conditions.

It is of interest to note the form of the Green's matrices for the case of Theorem 2:5. From the earlier discussion it is known that the Green's matrices may be defined on each subinterval $a_q a_{q+1}$, $(q = 1,2,\ldots, k-1)$, of ab by means of (2:36). For this particular problem, however, it is easier to derive the Green's matrix directly from its resolvent property. Let $Y(x)$ be a matrix whose columns form n linearly independent *continuous* solutions of the homogeneous differential equations of (2:2) on ab. Moreover, let a_o be an arbitrary point of this interval. Then

$$y^{(o)}(x) = \int_{a_o}^{x} Y(x)Y^{-1}(x_1)f(x_1)dx_1 \tag{2:53}$$

is a particular continuous solution of the differential equation

$$y' = A(x)y + f(x) \tag{2:54}$$

on ab, and the general continuous solution of this equation is

$$y(x) = y^{(o)}(x) + Y(x)c, \tag{2:55}$$

where c is an arbitrary constant vector (c_i). The homogeneous system consisting of the differential equations of (2:2), the boundary conditions (2:41), together with the condition that the solutions be continuous on the entire interval ab, is incompatible if and only if the matrix

$$D = \sum_{q=1}^{k} m_{:q} Y(a_q)$$

is non-singular. In this case the unique solution of the non-homogeneous system (2:54), (2:41) which is continuous on the entire interval ab is given by (2:55) with c determined as

$$c = -D^{-1}\sum_{q=1}^{k} \int_{a_o}^{a_q} m_{:q} Y(a_q) Y^{-1}(x_1) f(x_1) dx_1. \tag{2:56}$$

That is,

$$\begin{aligned} y(x) = &\int_{a_o}^{x} Y(x)Y^{-1}(x_1)f(x_1)dx_1 \\ &- \sum_{q=1}^{k}\int_{a_o}^{a_q} Y(x)D^{-1}m_{:q}Y(a_q)Y^{-1}(x_1)f(x_1)dx_1. \end{aligned} \tag{2:57}$$

The integrand functions in (2:57) are the Green's matrices

$$\mathcal{G}^{(q)}(x,x_1) = Y(x)D^{-1}m_{:q}Y(a_q)Y^{-1}(x_1), \qquad (q = 1,2,\ldots,k), \tag{2:58}$$

defined by Langer ([5]; (9.6)). As he has shown, these matrices are inter-related by the equations

$$\sum_{q=1}^{k} \mathcal{G}^{(q)}(x,x_1) = Y(x)Y^{-1}(x_1), \tag{2:59}$$

$$\mathcal{G}^{(q)}(x,a_q) = Y(x)D^{-1}m_{:q}, \tag{2:60}$$

$$(2:61) \quad \sum_{q=1}^{k} m_{:q}\, \mathcal{G}^{(p)}(a_q,x_1) = m_{:p}Y(a_p)Y^{-1}(x_1), \qquad (p = 1,2,\ldots,k).$$

3. Differential systems involving a parameter. Consider a system of the form

$$(3:1) \qquad \begin{aligned} y' &= (A + \lambda B)y, \\ s[y] &= 0, \end{aligned}$$

containing a characteristic parameter λ linearly, and having elements of the matrices $A(x)$, $B(x)$ continuous on ab, except possibly at $x = a_2, a_3, \ldots, a_{k-1}$. In general, however, it is assumed that the elements of $A(x)$ and $B(x)$ have finite right- and left-hand limits at these intermediate points. It is further supposed that the elements of the matrix $B(x)$ are not all identically zero on ab. The boundary conditions of (3:1) are of the form indicated in (2:1).

The transformations of Section 2 will carry (3:1) into a corresponding two-point system

$$(3:2) \qquad \begin{aligned} u' &= (\mathcal{A} + \lambda\mathcal{B})u, \\ s[u] &\equiv \mathcal{M}u(0) + \mathcal{N}u(1) = 0. \end{aligned}$$

The adjoint system of (3:1) is then found as before, and is written

$$(3:3) \qquad \begin{aligned} z' &= -z(A + \lambda B), \\ r[z] &= 0. \end{aligned}$$

It follows from the existence theorems for differential equations that there exists a matrix $\|Y_{ij:q}(x, \lambda)\|$ whose columns are n linearly independent solutions of the differential equations of (3:1) on each sub-interval $a_q < x < a_{q+1}$, and whose elements, together with their derivatives with respect to x, are expressible as permanently convergent power series in λ. It then follows that the determinant of the matrix $\mathcal{D}(\lambda)$ formed for this system is also a permanently convergent power series in λ. The zeros of $|\mathcal{D}(\lambda)|$ are called *characteristic values* of the parameter λ, and the non-trivial solutions $y(x)$ of (3:1) corresponding to such values are called *characteristic solutions*. Since $|\mathcal{D}(\lambda)|$ is a permanently convergent power series in λ, either this determinant is identically zero in λ or its zeros are at most denumerably infinite in number; moreover, if $|\mathcal{D}(\lambda)| \not\equiv 0$ the totality of its zeros has no finite accumulation points. We also have from Theorem 2:1 that the characteristic values for a problem (3:1) are identical with those for the adjoint problem (3:3), and that the number of linearly independent characteristic solutions is the same for both problems.

It is of particular interest in this section to investigate self-adjoint and definitely self-adjoint systems. In view of this, we shall define a k-point boundary value problem (3:1) to be self-adjoint if the transformed system (3:2), obtained from (3:1) by means of (2:3) and (2:4), is self-adjoint.

Using the terminology of Bliss [1], a system (3:2) is said to be self-adjoint if the differential equations and the boundary conditions of its adjoint

$$(3{:}4)\qquad \begin{aligned} v' &= -v(\mathcal{A} + \lambda\mathcal{B}), \\ r[v] &\equiv v(0)\mathcal{P} + v(1)\mathcal{Q} = 0, \end{aligned}$$

are individually equivalent to its own for all values of λ by means of a non-singular transformation $v(t) = \mathcal{T}(t)u(t)$, where the elements of $\mathcal{T}(t)$ are supposed to be continuous and have continuous derivatives on $0 \leqq t \leqq 1$. Bliss [1] has shown that necessary and sufficient conditions for (3:2) to be self-adjoint are

$$\mathcal{T}\mathcal{A} + \breve{\mathcal{A}}\mathcal{T} + \mathcal{T}' \equiv 0, \quad \mathcal{T}\mathcal{B} + \breve{\mathcal{B}}\mathcal{T} \equiv 0, \tag{3:5}$$

$$\mathcal{M}\,\mathcal{T}^{-1}(0)\breve{\mathcal{M}} = \mathcal{N}\mathcal{T}^{-1}(1)\breve{\mathcal{N}}. \tag{3:6}$$

Now it follows from a theorem of Reid [10] that the most general solution of the differential system of (3:5) is of the form

$$\mathcal{T}(t) = \breve{\mathcal{V}}(t)\mathcal{C}\mathcal{V}(t), \tag{3:7}$$

where $\mathcal{V}(t)$ is a fundamental matrix solution of the adjoint differential system $\mathcal{V}'(t) = -\mathcal{V}(t)\mathcal{A}(t)$ and $\mathcal{C}$ is a constant matrix. Moreover,

$$\mathcal{V}(t) \equiv \left\| \begin{matrix} Z^{(1)}(t) & 0 & \cdots & 0 \\ 0 & Z^{(2)}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Z^{(k-1)}(t) \end{matrix} \right\| \tag{3:8}$$

where $Z^{(q)}(t) \equiv \| Z_{ij:q}(x_q[t])\|$, and $Z_{:q}(x) \equiv \| Z_{ij:q}(x)\|$ is a fundamental solution of $Z'(x) = -Z(x)A(x)$ satisfying $Z_{:q}(x)Y_{:q}(x) = I$ on $a_q a_{q+1}$. In particular, the condition that $\mathcal{T}(t)$ is non-singular holds if and only if the matrix $\mathcal{C}$ is non-singular. Furthermore, $\mathcal{C}$ must satisfy the conditions

$$\breve{\mathcal{V}}(t)\mathcal{C}\mathcal{V}(t)\mathcal{B}(t) + \breve{\mathcal{B}}(t)\breve{\mathcal{V}}(t)\mathcal{C}\mathcal{V}(t) = 0, \tag{3:9}$$

$$\mathcal{M}\mathcal{V}^{-1}(0)\mathcal{C}^{-1}\breve{\mathcal{V}}^{-1}(0)\breve{\mathcal{M}} = \mathcal{N}\mathcal{V}^{-1}(1)\mathcal{C}^{-1}\breve{\mathcal{V}}^{-1}(1)\breve{\mathcal{N}}, \tag{3:10}$$

obtained by substituting (3:7) into the latter part of (3:5) and (3:6).

One might break the transformation matrix $\mathcal{T}(t)$ up into $(k-1)^2$ sub-matrices, each of n rows and columns, and transform conditions (3:5), (3:6) into conditions in terms of these component sub-matrices of $\mathcal{T}$ and the matrices $A(x)$, $B(x)$, $M_{:q}$, and $N_{:q}$ of system (3:1). However, since this procedure usually leads to a complication, rather than a simplification in the form of these statements, it seems preferable to speak of (3:5), (3:6) as the conditions for self-adjointness of both the two-point and the original k-point problem.

There is one case, however, in which the conditions (3:5), (3:6) may be expressed for the original k-point problem in a comparably simple form. Suppose $y(x)$ is a solution of a self-adjoint system (3:1), $u(t)$ the corresponding solution of (3:2), and $z(x)$ the solution of the adjoint system (3:3) corresponding to the solution $v(t) = \mathcal{T}(t)u(t)$. Then at a point x of the sub-interval $a_p < x < a_{p+1}$ the value of $z(x)$ is in general a linear form in the values of $y(x)$ at the $k-1$ points $x_q(t_p[x])$, $(q = 1,2,\dots, k-1)$. If, in particular, the value of $z(x)$ depends only upon the value of $y(x)$ at the one point $x_p(t_p[x]) = x$, the problem (3:1) is said to be strictly self-adjoint. In other words, if $y(x)$ is a solution of (3:1) such that $y(x) \equiv 0$ except on $a_p < x < a_{p+1}$, then the corresponding solution $z(x)$ of (3:3) is also identically zero except on the same sub-interval $a_p < x < a_{p+1}$.

It is easy to show that under these restrictions $\mathcal{T}$ must be a matrix of the form

$$(3:11) \qquad \mathcal{T}(t) \equiv \begin{Vmatrix} T^{(1)}(t) & 0 & \cdots & 0 \\ 0 & T^{(2)}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & T^{(k-1)}(t) \end{Vmatrix},$$

where $T^{(q)}(t)$, $(q = 1,2,\dots, k-1)$, are $n \times n$ matrices and 0 denotes the $n \times n$ zero matrix. The general solution of (3:5) is given by (3:7), so if $\mathcal{V}(t)$ is of the form (3:8), an obvious necessary and sufficient condition that (3:1) be strictly self-adjoint is that $\mathcal{C}$ shall be of the form

$$(3:12) \qquad \mathcal{C} \equiv \begin{Vmatrix} C^{(1)} & 0 & \dots & 0 \\ 0 & C^{(2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C^{(k-1)} \end{Vmatrix},$$

where $C^{(q)}$, $(q = 1,2,\dots, k-1)$, are non-singular constant matrices and 0 is again the n x n zero matrix.

THEOREM 3:1. *If* $\mathcal{V}(t)$ *is a particular fundamental matrix solution of* $\mathcal{V}' = -\mathcal{V}\mathcal{A}$ *of the form* (3:8), *then a necessary and sufficient condition that* (3:1) *be self-adjoint is that there exist a non-singular constant matrix* $\mathcal{C}$ *satisfying* (3:9) *and* (3:10). *In particular, if* (3:1) *is strictly self-adjoint,* $\mathcal{T}(t)$ *must be of the form* (3:11) *and* $\mathcal{C}$ *is also of the form* (3:12).

If a system (3:1) is strictly self-adjoint, then the above matrix $\mathcal{T}(t)$ defines a matrix

$$T(x) \equiv T^{(q)}(t_q[x]), \qquad a_q < x < a_{q+1}, \ (q = 1,2,\dots, k-1).$$

In general, T(x) has discontinuities at $a_2, a_3, \dots, a_{k-1}$, but the elements of T(x) have finite right- and left-hand limits at these points. In terms of T(x) conditions (3:5), (3:6) may be stated for the original system (3:1) on the interval ab as follows:

$$(3:5') \qquad TA + \breve{A}T + T' \equiv 0, \qquad TB + \breve{B}T \equiv 0,$$

$$(3:6') \qquad \sum_{q=1}^{k-1} M_{:q}T^{-1}(a_q{}^{+})\breve{M}_{:q} = \sum_{q=1}^{k-1} N_{:q}T^{-1}(a_{q+1}^{-})\breve{N}_{:q}.$$

In the consideration of definitely self-adjoint boundary value problems the elements of the matrices A(x), B(x), M, N are taken to be real on the interval ab. A boundary problem (3:2) is said to be *definitely self-adjoint* (Bliss [2]) if it is self-adjoint with respect to a matrix $\mathcal{T}(t)$ and has the further properties:

(1) *the matrix of functions* $\mathcal{S}(t) = \breve{\mathcal{T}}(t)\mathcal{B}(t)$ *is symmetric on* $0 \leqq t \leqq 1$;

(2) *the quadratic form* $\xi\mathcal{S}(t)\xi$ *is positive semi-definite on* $0 \leqq t \leqq 1$;

(3) $u(t) \equiv 0$ *is the only vector which satisfies on* $0 \leqq t \leqq 1$ *the conditions*

$$(3{:}13) \qquad u' = \mathcal{A}u, \qquad s[u] = 0, \qquad u\mathcal{S}u \equiv 0.$$

The system (3:1) will be called *definitely self-adjoint* in case the transformed system (3:2) is definitely self-adjoint in the sense of Bliss [2].

Theorem 3:1 gives necessary and sufficient conditions for self-adjointness of (3:1). Hence, we have only to add the properties (1), (2) and (3) above to obtain necessary and sufficient conditions that (3:1) shall be definitely self-adjoint. Again, there is in general no gain in simplicity by writing these conditions in terms of the coefficients of (3:1).

A boundary value problem (3:1) is said to be *strictly definitely self-adjoint* with a matrix $T(x)$ if in addition to being strictly self-adjoint with respect to this matrix, the above conditions (1), (2), and (3) hold for the corresponding matrix $\mathcal{T}(t)$ given by (3:11). In view of the special form of $\mathcal{T}(t)$, $\mathcal{A}(t)$, and $\mathcal{B}(t)$, these latter conditions become

(1') *the matrix* $S(x) = \breve{T}(x)B(x)$ *is symmetric at each value* x *on* ab;

(2') *the quadratic form* $wS(x)w$ *is positive semi-definite on* ab;

(3') $y(x) \equiv 0$ *is the only vector which satisfies on* ab

the conditions

$$(3:14) \qquad y' = Ay, \qquad s[y] = 0, \qquad ySy \equiv 0.$$

Many interesting results have been obtained, (Bliss [2]), for definitely self-adjoint systems with two-point boundary conditions. For such a system, the characteristic values are all real and the corresponding characteristic solutions of the boundary value problem can be chosen real. The index of each characteristic value is equal to its multiplicity. Some very general expansion theorems in terms of the characteristic solutions also hold for such a system, (Bliss [2], pp. 414-419). The results for the two-point case clearly imply corresponding results for a k-point problem that is definitely self-adjoint as defined above.

Some further questions may arise when the elements of the coefficient matrices are complex valued functions. For this case, the system

$$(3:15) \qquad \begin{aligned} \mu'(t) &= -\mu(t)[\bar{\mathcal{A}}(t) + \lambda \bar{\mathcal{B}}(t)], \\ \bar{\varrho}[\mu] &\equiv \mu(0)\bar{\mathcal{P}} + \mu(1)\bar{\mathcal{Q}} = 0, \end{aligned}$$

will be called the conjugate of (3:4) and the conjugate adjoint of (3:2). Further, the system (3:2) will be called self-conjugate adjoint if it is equivalent to (3:15) under a non-singular linear transformation $\mu(t) = \mathcal{T}(t)u(t)$, where the elements of $\mathcal{T}$ are allowed to be complex valued functions which are continuous and have continuous derivatives on $0 \leqq t \leqq 1$, (Reid [10]). Of course, in the case where $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ are real, the elements of $\mathcal{P}, \mathcal{Q}$ may be chosen real, so that (3:14) is then identical with (3:4). If $\mu = \mu(t)$ is a solution of (3:15) for a value of λ, then $u = \bar{\mu}(t)$ is a solution of (3:2) for the corresponding conjugate value $\bar{\lambda}$.

Corresponding to the definition of definitely self-adjoint systems with real coefficients given by Bliss [2], Reid [10] has defined the system (3:2) to be *definitely self-conjugate adjoint* if it is self-conjugate adjoint with respect to $\mathcal{T}(t)$, and has the further properties:

(1") *the matrix* $\mathcal{S}(t) \equiv \mathcal{T}^*(t)\mathcal{B}(t)$ *is hermitian on* $0 \leqq t \leqq 1$;

(2") *the hermitian form* $\bar{\xi}\mathcal{S}(t)\xi$ *is positive semi-definite on* $0 \leqq t \leqq 1$;

(3") $u(t) \equiv 0$ *is the only vector which satisfies on* $0 \leqq t \leqq 1$ *the conditions*

$$(3{:}16) \qquad u' = \mathcal{A}u, \qquad s[u] = 0, \quad \bar{u}\mathcal{S}(t)u \equiv 0.$$

Since all the characteristic values of a definitely self-conjugate adjoint system (3:2) are real (Reid [10], Theorem 3:1), the consideration of the existence of characteristic values and related expansion theorems for this type of system is reducible to the analogous consideration for an associated definitely self-adjoint system with real coefficients.

A boundary problem (3:1) may be said to be *strictly definitely self-conjugate adjoint* with a matrix T(x) if in addition to being strictly definitely self-adjoint, the above conditions (1"), (2") and (3") hold for the corresponding matrix $\mathcal{T}(t)$.

4. *A special boundary value problem of the calculus of variations.* The theory of differential systems with k-point boundary conditions, as discussed in Sections 2 and 3, can be applied to special differential systems associated with the calculus of variations. Such special systems have been considered in previous investigations. The accessory boundary value problem associated with a minimizing extremaloid for a problem of the

calculus of variations is of this type (Reid [7]). Denbow [3] considered a calculus of variations problem of Bolza type in which the integrand function involved not only the coordinates of the end points of an admissible curve, but also the coordinates of variable intermediate points along this curve. The accessory boundary value problem for such a calculus of variations problem may also be written in the form discussed in the preceding section. The type of differential system involved in these problems may be transformed, by the use of transformations discussed in Section 2, to obtain a differential system with only two-point boundary conditions. Furthermore, the transformed problem may be written in the form presented by Hu [4]. This form is of particular interest, because Hu has given _necessary_ and _sufficient_ conditions for such differential systems to arise from certain problems of the calculus of variations. We may then ask for the corresponding conditions under which the k-point differential systems will arise from problems of the calculus of variations.

The system that Hu considered is of the form

$$\begin{aligned} \eta_i'(x) &= L_{ij}(x)\,\eta_j(x) + M_{ij}(x)\,\zeta_j(x), \\ \zeta_i'(x) &= N_{ij}(x)\,\eta_j(x) - L_{ji}(x)\,\zeta_j(x) - \lambda K_{ij}(x)\,\eta_j(x), \end{aligned} \tag{4:1}$$

$$\begin{gathered} s_\rho[\eta,\zeta] \equiv a^1_{\rho j}\,\eta_j(x_1) - b^1_{\rho j}\,\zeta_j(x_1) + a^2_{\rho j}\,\eta_j(x_k) + b^2_{\rho j}\,\zeta_j(x_k) = 0, \\ (i,j = 1,2,\dots,n;\ \rho = 1,2,\dots,2n), \end{gathered} \tag{4:2}$$

with K the identity matrix. For the purpose of present consideration, we will assume $K(x)$ positive definite on x_1x_k. Hu also made use of the two following fundamental properties.

PROPERTY 1. _The matrices_ $\|M_{ij}(x)\|$ _and_ $\|N_{ij}(x)\|$ _are symmetric, and the former has constant rank_ $n-m$ _on the interval_ x_1x_k.

PROPERTY 2. *The* 2n x n *matrices* $\| a^{1}_{\rho j} \|$, $\| b^{1}_{\rho j} \|$, $\| a^{2}_{\rho j} \|$, $\| b^{2}_{\rho j} \|$ *are such that the* 2n x 2n *matrix*

$$\| a^{1}\breve{b}^{1} + a^{2}\breve{b}^{2} \| \tag{4:3}$$

is symmetric.

Indeed, Hu has shown that Properties 1 and 2 are necessary and sufficient conditions for (4:1) (4:2) to be the canonical form of the Euler-Lagrange differential equations and transversality conditions for a quadratic functional of the form

$$2H\,[\eta(x_1),\ \eta(x_k)] + \int_{x_1}^{x_k} 2\,\omega(x, \eta,\ \eta')dx \tag{4:4}$$

subject to a set of linear homogeneous differential equations of the form

$$\Phi_{\alpha}(x, \eta,\ \eta') \equiv \Phi_{\alpha\eta'_j}(x)\eta'_j + \Phi_{\alpha\eta_j}(x)\eta_j = 0, \quad (\alpha = 1,2,\ldots,\ m < n), \tag{4:5}$$

a set of linear homogeneous boundary conditions

$$\Psi_{\gamma}[\eta] \equiv \Psi_{\gamma:j1}\,\eta_j(x_1) + \Psi_{\gamma:jk}\,\eta_j(x_k) = 0, \quad (\gamma = 1,2,\ldots,\ p \leqq 2n), \tag{4:6}$$

and the isoperimetric norming condition

$$\int_{x_1}^{x_k} \eta_i K_{ij} \eta_j dx = 1. \tag{4:7}$$

In the above, η denotes a set of functions $[\eta_i(x)]$, $(i = 1,2,\ldots,n)$, while η' represents the corresponding set of derivatives $[\eta'_i(x)]$. In (4:4), 2H is a quadratic form in the 2n values $\eta_i(x_1)$, $\eta_i(x_k)$, and 2ω is a quadratic form in η_i, η'_i which may be written specifically as

$$2\omega\,(x, \eta\,, \eta') \equiv \omega_{\eta'_i\eta'_j}(x)\,\eta'_i\eta'_j + 2\,\omega_{\eta'_i\eta_j}(x)\,\eta'_i\eta_j + \omega_{\eta_i\eta_j}(x)\,\eta_i\eta_j. \tag{4:8}$$

Now consider a k-point problem consisting of differential equations of the form (4:1) and boundary conditions

$$s_\sigma[\eta,\zeta] \equiv \sum_{q=1}^{k-1}\sum_{j=1}^{n}[a^1_{\sigma j:q}\eta_j(x_q^+) - b^1_{\sigma j:q}\zeta_j(x_q^+) + a^2_{\sigma j:q}\eta_j(x_{q+1}^-) + b^2_{\sigma j:q}\zeta_j(x_{q+1}^-)] = 0, \tag{4:9}$$

$$(\sigma = 1,2,\ldots,\ \nu = (k-1)2n),$$

where $x_1 < x_2 < \ldots < x_k$. We shall assume that the functions L_{ij}, M_{ij}, N_{ij} and K_{ij} are continuous on x_1x_k, except possibly at the intermediate points $x_2, x_3, \ldots, x_{k-1}$, where it is merely assumed that these functions have finite right- and left-hand limits. We shall assume throughout the following discussion of this k-point problem that it is normal; that is, for it there exists no non-trivial solutions η_i, ζ_i with $\eta_i \equiv 0$ on x_1x_k. We shall now consider the question of necessary and sufficient conditions for a system (4:1), (4:9) to be the canonical form of the Euler-Lagrange differential equations and transversality conditions for a quadratic functional

$$2H[\eta(x_1^+),\ldots,\eta(x_{k-1}^+),\ \eta(x_2^-),\ldots,\eta(x_k^-)] + \int_{x_1}^{x_k} 2\omega(x,\eta,\eta')dx, \tag{4:10}$$

subject on each sub-interval $x_q < x < x_{q+1}$ to a corresponding set of linear differential equations

$$\Phi_{\alpha:q}(x,\eta,\eta') \equiv \Phi_{\alpha\eta'_j:q}(x)\eta'_j + \Phi_{\alpha\eta_j:q}(x)\eta_j = 0, \tag{4:11}$$

$$(\alpha = 1,2,\ldots,m_q < n;\ q = 1,2,\ldots,k-1),$$

a set of linear homogeneous boundary conditions

$$\Psi_\gamma[\eta(x_1^+),\ldots,\eta(x_{k-1}^+),\ \eta(x_2^-),\ldots,\eta(x_k^-)] = 0, \tag{4:12}$$

$$(\gamma = 1,2,\ldots,p \leqq \nu),$$

and the isoperimetric norming condition

$$(4:13) \qquad \int_{x_1}^{x_k} \eta_i K_{ij} \eta_j dx = 1.$$

Denbow has shown ([3], Section 2) that this k-point calculus of variations problem may be transformed into a two-point problem of the type considered by Hu [4] ; as a matter of fact, the transformations that we have employed in Section 2 are identical with those of Denbow. Moreover, such a transformation of this calculus of variations problem induces on the canonical form of the Euler-Lagrange differential equations the transversality conditions exactly the type of change as discussed in Section 2 for the k-point problem (2:1). In particular, if $\| M_{ij} \|$, $\| N_{ij} \|$ and $\| K_{ij} \|$ are symmetric, then in the related two-point problem the corresponding matrices are also symmetric; if $\| M_{ij} \|$ is of constant rank $n - m_q$ on each sub-interval, then the corresponding matrix for the two-point problem has constant rank $n(k-1) - m_1 - m_2 - \ldots - m_{k-1}$ on $0 \leqq t \leqq 1$. It is thus possible to arrive at a two-point boundary problem of the form (4:1), (4:2) in $\nu = 2n(k-1)$ variables for which Hu's Properties 1 and 2 are necessary and sufficient conditions for this system to arise from a problem in the calculus of variations. Since we have shown the equivalence of the k-point and two-point boundary problems in Section 2, it is now possible to state for the k-point system (4:1), (4:9) the following analogues of the above Properties 1 and 2.

Property 1'. *The matrices* $\| M_{ij}(x) \|$, $\| N_{ij}(x) \|$ *and* $\| K_{ij}(x) \|$ *are symmetric, and* $\| M_{ij}(x) \|$ *has constant rank* $n - m_q$ *on each sub-interval* $x_q \leqq x \leqq x_{q+1}$, $(q = 1, 2, \ldots, k-1)$.

Property 2'. *The* $\nu \times n$ *matrices* $\| a^1_{:q} \|$, $\| b^1_{:q} \|$, $\| a^2_{:q} \|$ *and* $\| b^2_{:q} \|$ *of the boundary conditions* (4:9) *are such that the* $2\nu \times 2\nu$ *matrix*

$$(4{:}14) \qquad \mathfrak{A} \equiv \sum_{q=1}^{k-1} \| a^1_{:q} \check{b}^1_{:q} + a^2_{:q} \check{b}^2_{:q} \|$$

is symmetric.

Properties 1' and 2' are, consequently, necessary and sufficient conditions for the k-point system (4:1), (4:9) to be the canonical form of the Euler-Lagrange differential equations and transversality conditions for a calculus of variations problem of the form defined by (4:10), (4:11), (4:12) and (4:13). In particular, when K is the identity matrix we obtain a problem corresponding to that which Hu [4] discussed, and for which he proved the existence of infinitely many characteristic values.

Reid [9] has shown that a two-point system (4:1), (4:2) has a denumerable infinity of characteristic values under a weaker condition than that assumed by Hu. If we apply Reid's result to the two-point problem into which (4:1), (4:9) is transformed, it follows that a system (4:1), (4:9) will certainly have infinitely many characteristic values if, in addition to the symmetry of $\| M_{ij}(x) \|$, $\| N_{ij}(x) \|$, $\| K_{ij}(x) \|$ and Property 2', we have merely that not all the elements of $\| M_{ij} \|$ are identically zero on $x_1 x_k$.

By direct computation it may be readily verified that a system (4:1), (4:9) is strictly definite self-adjoint with respect to the constant matrix

$$(4{:}15) \qquad T \equiv \left\| \begin{matrix} 0 & I \\ -I & 0 \end{matrix} \right\| ,$$

where I and 0 are, respectively, the n x n identity and zero matrices. For this problem we have that the matrix S(x) is equal to

$$(4:16) \qquad S(x) \equiv \begin{Vmatrix} 0 & -I \\ I & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 \\ -K & 0 \end{Vmatrix} = \begin{Vmatrix} K & 0 \\ 0 & 0 \end{Vmatrix}.$$

5. Boundary value problems in the complex plane. Langer [5] has extended to two-dimensional regions of the complex plane various general results on linear boundary problems which were previously obtained only for an interval of the real axis. We shall now obtain some of the results of Langer by applying the methods of Section 2. More specifically, we shall establish the form of the Green's matrix for a boundary problem whose linear side conditions apply at k distinct points of the complex plane.

Consider the boundary value problem in the complex plane consisting of the differential equations and side conditions

$$(5:1) \qquad \begin{aligned} w_i'(\xi) &= \sum_{j=1}^{n} A_{ij}(\xi)w_j(\xi), \\ s_i[w] &\equiv \sum_{q=1}^{k}\sum_{j=1}^{n} m_{ij:q}w_j(\xi_q) = 0, \qquad (i=1,2,\ldots,n). \end{aligned}$$

The $A_{ij}(\xi)$ are supposed to be single-valued analytic functions in a simply connected region S of the complex $\xi = \eta + \sqrt{-1}\,\zeta$ plane, and $\xi_1, \xi_2, \ldots, \xi_k$ are k given distinct values in this region.

By a solution of (5:1) we shall understand a vector $w(\xi)$ whose elements are single-valued and analytic in S, and whose values at these prescribed points satisfy the linear homogeneous conditions $s_i[w] = 0$. Now, by the fundamental existence theorem for differential equations in the complex plane, there exists a fundamental matrix of solutions $W(\xi) \equiv \| W_{ij}(\xi) \|$ for the differential equations of (5:1) in the simply connected region S; moreover, the elements of $W(\xi)$ are single-valued analytic functions in this region (see, for example,[11] , pp. 139 ff. and 143 ff.).

Clearly the differential system (5:1) is compatible if and only if the matrix

$$(5:2)\quad D \equiv \| D_{ij} \| \equiv \| \sum_{q=1}^{k} \sum_{h=1}^{n} m_{ih:q} w_{hj}(\xi_q) \|, \qquad (i = 1,2,\ldots,n).$$

is singular.

Let

$$(5:3)\quad \begin{array}{c} C_q: \quad \xi = X(x) = X_1(x) + \sqrt{-1}\, X_2(x), \\ (q-1 \leqq x \leqq q;\ q = 1,2,\ldots,\ k-1), \end{array}$$

be the equation of a simple arc which lies in S and joins ξ_q to ξ_{q+1}, and whose defining functions $X_1(x)$ and $X_2(x)$ are continuous and have continuous derivatives satisfying $|X'(x)| \neq 0$. In order to have our notation correspond to that of Section 2, set $a_q = q - 1$, $(q = 1,2,\ldots,k)$; then

$$(5:4)\qquad C: \quad \xi = X(x), \qquad (a_1 \leqq x \leqq a_k),$$

is the equation of a path curve lying in S; moreover, since

$$(5:5)\qquad X(a_q) = \xi_q, \qquad (q = 1,2,\ldots,k),$$

this curve passes successively through the points $\xi_1, \xi_2, \ldots, \xi_k$ at which the boundary conditions of (5:1) apply. Clearly, one may require that any or all of the component arcs C_q of C pass through a given point ξ of S. If corresponding to a solution $w \equiv (w_i)$ of the differential equations of (5:1) we set

$$(5:6)\qquad y_i(x) \equiv w_i[X(x)], \qquad (a_1 \leqq x \leqq a_k),$$

we have

$$(5:7)\qquad y_i(a_q) = w_i(\xi_q).$$

Moreover, except possibly at the intermediate points $a_2, a_3, \ldots, a_{k-1}$ the derivative dy_i/dx exists and is given by

$$\frac{d}{dx} y_i(x) = w_i'[X(x)] X'(x). \tag{5:8}$$

Hence y(x) satisfies the differential equations and boundary conditions

$$\begin{aligned} y_i'(x) &= \sum_{j=1}^{n} X'(x) A_{ij}[X(x)] y_j(x), \\ s_i[y] &\equiv \sum_{j=1}^{n} \sum_{q=1}^{k} m_{ij:q} y_j(a_q) = 0, \\ &(i = 1,2,\ldots,n), \end{aligned} \tag{5:9}$$

and is further restricted by the conditions that the components of y(x) are continuous across the intermediate points $a_2, a_3, \ldots, a_{k-1}$. Corresponding to the fundamental matrix solution $W(\xi)$ of the differential equations of (5:1) in S, it is readily seen that $Y(x) \equiv W[X(x)]$ is a fundamental matrix solution of the differential equations of (5:9) along C; furthermore, the elements of Y(x) are continuous on the entire interval $a_1 \leqq x \leqq a_k$.

Comparing with Section 2, (5:9) is seen to be of the form (2:2) with boundary conditions (2:41). Consequently, Theorem 2:5 affords an alternative representation of the adjoint boundary conditions of the type discussed by Langer. Furthermore, the Green's matrices for (5:9) may be taken in the form (2:58). Inverting the transformations (5:5) - (5:8) and utilizing the above mentioned fact that the curve C may be required to pass through any prescribed point of S, it is then found that the form of the adjoint system and the Green's matrices are identical with those obtained by Langer [5] in the complex domain.

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AN ANALOGUE OF GREEN'S THEOREM FOR MULTIPLE INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS

BY

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AN ANALOGUE OF GREEN'S THEOREM FOR MULTIPLE INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS

1. Introduction. The problem to be considered is that of finding necessary and sufficient conditions for the following equation to hold

$$(1.1)\quad \int_A [v_\alpha(x)\eta_{x_\alpha}(x) + w(x)\eta(x)]dA = \int_B \eta(x)v_\alpha(x)\lambda_\alpha \, dB, \qquad (\alpha = 1, \dots, m)$$

where the integral on the left is taken over a bounded region A of points $(x) = (x_1, \dots, x_m)$ and that on the right is taken over the boundary B of A. The symbol λ_α represents the direction cosines of the outer normal to B. By the value of an integral of the form $\int f(x)dB$ over a piece of the boundary B defined by functions $x_i(t)$, (t on T), will be meant the integral $\int_T f(x(t))\sqrt{c_\alpha c_\alpha}\, dt_1, \dots, dt_{m-1}$, where c_α is the cofactor of a_α in the determinant $|a_\alpha \frac{\partial x_\alpha}{\partial t_\gamma}|$, $(\gamma = 1, \dots, m - 1)$. The method used in this paper is essentially a generalization and extension of that used by Hestenes [XII]* in the two-dimensional case. The proofs here given have the advantage that no assumptions need to be made on the behavior of the functions v_α outside of the region A + B. Moreover, no assumptions need to be made regarding the intersections of the boundary with hyperplanes parallel to the coordinate hyperplanes.

Among other things it is shown that equation (1.1) holds if and only if the relation

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*Roman numerals in brackets refer to the bibliography at the end of this paper.

$$\int_{A'} w(x)\,dA = \int_{B'} v_\alpha(x)\chi_\alpha\,dB \tag{1.2}$$

is satisfied for every m-dimensional cube A' whose closure is in A and whose sides are parallel to the coordinate hyperplanes. This is done under very general hypotheses concerning the boundary B and the functions $v_\alpha(x)$ and $w(x)$. If the function $w(x)$ is continuous and the functions $v_\alpha(x)$ have continuous derivatives then equation (1.2) is equivalent to the equation $w = v_{\alpha x_\alpha}$. Setting $\eta = 1$ and $w = v_{\alpha x_\alpha}$ in equation (1.1), one obtains Green's formula

$$\int_A v_{\alpha x_\alpha}\,dA = \int_B v_\alpha \chi_\alpha\,dB.$$

Thus equation (1.1) can be considered to be an analogue of Green's formula. Since our proofs are made independently of Green's theorem, they constitute a proof of this theorem for m-dimensional space.

Equation (1.1) arises in the study of the first variation of a multiple integral of the form

$$\int_A f\left[x_1, \ldots, x_m, y_1(x), \ldots, y_n(x), \frac{\partial y_1}{\partial x_1}, \ldots, \frac{\partial y_n}{\partial x_m}\right] dA.$$

By means of this equation and, in particular, equation (1.2), one obtains the Haar-Coral equations

$$\int_A f_{y_i}\,dA = \int_B f_{y_{ix_\alpha}}\chi_\alpha\,dB, \qquad (i = 1, \ldots, n)$$

edge conditions, and an analogue of Jacobi's condition for a minimizing manifold of class c'.

After formulating the problem in Section 2, we proceed in Section 3 to show that equation (1.1) holds for all admissible functions if and only if it holds for certain special classes of admissible functions. Section 4 is devoted to the proof of Green's Theorem and other associated results. The previous results are generalized further in Sections 5 and 6 by considering a more general type of boundary B and discussing the case in which $v_\alpha(x)$ have certain types of discontinuities. In Section 8, some of the previous results are applied in proving theorems which are useful in mechanics. The remainder of the paper is concerned with applications to the problem of the calculus of variations for multiple integrals and includes the derivation of Haar-Coral equations, edge conditions, and an analogue of Jacobi's condition.

2. Analytic formulation of the problem. In Sections 3 and 4 a region A of points $(x) \equiv (x_1, \ldots, x_m)$ will be called an admissible region in case it is bounded and its boundary B is non-singular and of class c'. Then the boundary B can be represented locally by equations of the form $x_\alpha = x_\alpha(t_1, \ldots, t_{m-1})$, $(\alpha = 1, \ldots, m)$, where the functions $x_\alpha(t)$ are of class c' and the matrix $\left\| \frac{\partial x_\alpha}{\partial t_\gamma} \right\|$, $(\gamma = 1, \ldots, m - 1)$, is of rank m - 1. A function will be said to be of class c' on A + B in case it is continuous and has a continuous differential on A + B.

By an admissible function $\eta(x)$ will be meant one belonging to a class $\mathcal{H}$ of continuous functions on A + B having the following properties: (1) Every non-negative function of class c^∞ on A + B is in $\mathcal{H}$; (2) The product of a function in $\mathcal{H}$ by a non-negative function of class c^∞ on A + B is in $\mathcal{H}$; (3) On each interval $a_\alpha \leqq x_\alpha \leqq b_\alpha$ in A, a function $\eta(x)$ in $\mathcal{H}$ is absolutely continuous in each variable $x_1, \ldots, x_m$ when the remaining variables are fixed and the derivatives $\pi_\alpha \equiv \eta_{x_\alpha}$ are integrable on A.

The smallest class of admissible functions is the class H_1 of all non-negative functions of class c^∞ on A + B. The largest class is the class H_2 of all functions having property (3). Any subclass of H_2 containing H_1 and having property (2) is admissible. The class commonly used in the calculus of variations is the class of all functions of class c' on A + B. These functions form a class of admissible functions.

Let $v_1(x), \ldots, v_m(x)$ be m real valued functions which are continuous on A + B, and let $w(x)$ denote a real single valued function that is integrable on A in the sense of Lebesgue.

We shall find necessary and sufficient conditions for the following equation to hold

$$\int_A (v_\alpha \pi_\alpha + w\eta)\,dA = \int_B \eta\, v_\alpha \lambda_\alpha \, dB, \tag{2.1}$$

where $\lambda_1, \ldots, \lambda_m$ are the direction cosines of the outer normal to B. Here and elsewhere the repetition of a subscript will denote summation unless otherwise specified.

In Section 5, the class of admissible regions will be enlarged and in Section 6, we shall discuss the case where the functions v_α have certain types of discontinuities. We restrict ourselves to the case here described in order to simplify the argument.

In Section 3, the following result will be used.

Lemma 2.1. *The form of equation (2.1) is unaltered by a rotation or translation of axes.*

Since the form of equation (2.1) is obviously unchanged by a translation, consider a rotation $x_\alpha = A_{\alpha\beta} z_\beta$, $(\alpha, \beta = 1, \ldots, m)$. Let $y(z)$ represent the function into which $\eta(x)$ is transformed by this rotation. The derivatives $p_\alpha \equiv y_{z_\alpha}$ and $\pi_\alpha \equiv \eta_{x_\alpha}$ are connected by the formula $\pi_\alpha = A_{\alpha\beta} p_\beta$. If we set $V_\beta \equiv v_\alpha A_{\alpha\beta}$ then $V_\beta p_\beta = v_\alpha \pi_\alpha$. Moreover, the direction cosines of the normal to B are related by the formula $\lambda_\alpha = A_{\alpha\beta} L_\beta$; where the symbol L_β represents the new direction cosines of the normal to the boundary B. Thus, we see that $v_\alpha \lambda_\alpha = v_\alpha (A_{\alpha\beta} L_\beta) = V_\beta L_\beta$ and integral (1.1) takes the form

$$\int_A (V_\beta p_\beta + wy)\,dA = \int_B yV_\beta L_\beta\,dB.$$

This completes the proof of Lemma 2.1.

3. *Preliminary Theorems*. In this section it will be shown that equation (2.1) holds for all admissible functions $\eta(x)$ if and only if it holds for certain special classes of admissible functions. As a first result, we have

Theorem 3.1. *The equation* (2.1) *holds for every admissible function* $\eta(x)$ *if and only if to each point* P *on* A + B *there is a neighborhood* N *of* P *such that equation* (2.1) *is true for every admissible function* $\eta(x)$ *which is identically zero on* A - AN.

In order to prove this, cover A + B with a finite number of spheres $K_1, K_2, \ldots, K_q$ with centers $P_1, \ldots, P_q$ on A + B and radii $r_1, \ldots, r_q$ such that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on $A - AK_i'$, where K_i' is a sphere of radius $2r_i$ about the point P_i. Let $d_i(x)$ be the distance from the point (x) to the point P_i and set $h_i(x) = h(\frac{d_i(x)}{r_i})$, where h(t) is defined to be a function of class c^∞ and such that for $t \leqq 1$, $h(t) = 0$; for $t \geqq 2$, $h(t) = 1$; and $h'(t) \geqq 0$. A function of this type used by Hestenes [XII] in a similar situation is the function h(t) having $h(t) = 0$ for $t \leqq 1$; $h(t) = 1$ for $t \geqq 2$; and $h(t) = e^{g(t)/1-t}$ for $1 < t < 2$, where $g(t) = e^{1/t-2}$.

Let $\eta(x)$ be any admissible function and consider the functions

$$\begin{aligned} \eta_1 &= (1 - h_1)\eta, \\ \eta_2 &= h_1(1 - h_2)\eta, \\ &\cdot \quad \cdots \\ \eta_q &= h_1h_2, \ldots, h_{q-1}(1 - h_q)\eta. \end{aligned} \tag{3.1}$$

Adding, we obtain $\eta = \eta_1 + \eta_2 + \cdots + \eta_q + h_1 h_2 \ldots h_q \eta$. Since the function h_i vanishes on sphere K_i and the spheres $K_1, \ldots, K_q$ cover A + B, the product $h_1 h_2 \ldots h_q$ is identically zero on A + B and we have $\eta = \eta_1 + \eta_2 + \cdots + \eta_q$ on A + B. Let

$$L(\eta) = \int_A (v \pi_\alpha + w\eta)\, dA - \int_B \eta v_\alpha \gamma_\alpha \, dB.$$

Since the functions η_i vanish on $A - AK_i'$, it follows that $L(\eta_i) = 0$. Due to the linearity of $L(\eta)$, we have

$$
\begin{aligned}
(3.2) \qquad L(\eta) &= L(\eta_1 + \eta_2 + \cdots + \eta_q) \\
&= L(\eta_1) + L(\eta_2) + \cdots + L(\eta_q) = 0
\end{aligned}
$$

as was to be proved.

Theorem 3.2. *Equation (2.1) holds for every admissible function $\eta(x)$ if and only if it holds for every admissible function $\eta(x)$ which vanishes on a neighborhood of the boundary.*

Using the criterion that equation (2.1) holds for every admissible function $\eta(x)$ vanishing on a neighborhood of the boundary, we know that to each point P of A there is a neighborhood N of P such that $L(\eta) = 0$ for every admissible function $\eta(x)$ which vanishes on A - AN. In order to complete the proof by use of Theorem 3.1 we will show that points of B have the same property.

By Lemma 2.1, we can assume that the boundary point P under consideration is the origin and that the positive x_m-axis is the inner normal of B at P. We define a neighborhood N of P by the inequalities

$$(3.3) \qquad -e < x_\gamma < e, \quad -e' < x_m < e' \quad (\gamma = 1, \ldots, m-1),$$

where e, e' are so small that the boundary of B in N can be defined in this neighborhood by an equation of the form

$x_m = f(x_1, x_2, \ldots, x_{m-1})$, the function f being single-valued and of class c' in the region defined by inequalities (3.3). Let

$$H(x, r) = h\left(\frac{x_m - f}{r}\right)$$

where r is an arbitrary positive parameter and h(t) is the function used in proof of Theorem 3.1 above. We have $H = 0$ when $x_m \leqq f + r$ and $H = 1$ when $x_m \geqq f + 2r$. Consider an admissible function $\eta(x)$ vanishing on A - AN and let ζ be equal to $H\eta$ on N and identically zero elsewhere. The function ζ is admissible and vanishes at all points of N for which $x_m \leqq f + r$. Therefore, ζ vanishes in a neighborhood of the boundary B, so that $L(\zeta) = 0$ by virtue of our hypothesis. Thus we have

$$\begin{aligned} 0 = L(\zeta) &= \int_{AN} (v_\alpha \zeta_{x_\alpha} + w\zeta)\,dA - \int_{BN} \zeta v_\alpha \lambda_\alpha \, dB \\ &= \int_{AN} H(v_\alpha \pi_\alpha + w\eta)\,dA + \int_{AN} \eta v_\alpha H_{x_\alpha} \, dA \end{aligned}$$

from which we find that

$$\int_{AN} H(v_\alpha \pi_\alpha + w\eta)\,dA = -\int_{AN} \eta v_\alpha H_{x_\alpha} \, dA.$$

The first member of this equation has the first member of equation (2.1) as its limit since H approaches 1 when r tends to zero. It will be shown that the remaining part of this equation has the second member of equation (2.1) as its limit. In this connection, we note that $H_{x_\alpha} = 0$ when $x_m \geqq f + 2r$. Taking r sufficiently small and setting $x_m = f + rt$, we have

$$-\int_{AN} \eta(x) v_\alpha(x) H_{x_\alpha} \, dA = \int_{AN} \eta(X) v_\alpha(X) h'(t) \lambda_\alpha \frac{\sqrt{1 + f_{x_\beta} f_{x_\beta}}}{r} \, dA$$

where $X = (x_1, \ldots, x_{m-1}, f + rt)$, $(\beta = 1, \ldots, m)$, and the symbol λ_α represents the direction cosines of the outer normal to the boundary B. The integral may then be written in the form

$$\int_{BN} dB \int_0^2 \eta(x) v_\alpha(x) \chi_\alpha h'(t)dt.$$

Taking the limit as r approaches zero, the result is

$$\int_{BN} \eta(x) v_\alpha(x) \chi_\alpha \, dB \int_0^2 h'(t)dt = \int_B \eta(x) v_\alpha(x) \chi_\alpha \, dB$$

since η vanishes outside BN. This completes the proof of Theorem 3.2.

Theorem 3.3. *The equation (2.1) holds for every admissible function $\eta(x)$ if and only if to each point P of A there is a neighborhood N of P whose closure is in A and such that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on A - AN.*

In proof of this result, let $\eta(x)$ be an admissible function vanishing on a neighborhood N of boundary B. Proceeding as in the proof of Theorem 3.1 with A + B replaced by A - AN, cover the bounded closed set A - AN by a finite number of spheres $K_1, \ldots, K_q$ with centers $P_1, \ldots, P_q$ on A and radii $r_1, \ldots, r_q$. These spheres can be chosen such that for each i the sphere K_i' of radius $2r_i$ about point P_i lies interior to A and is such that equation (2.1) holds for every admissible function which vanishes on A - AK_i. Construct functions $h_i(x)$ related to the spheres K_i and K_i', as in the proof of Theorem 3.1. Let $\eta_1, \ldots, \eta_q$ be given by the formulas (3.1). Then $\eta = \eta_1 + \cdots + \eta_q$ on A - AN. Since, however, η and hence $\eta_1, \ldots, \eta_q$ are zero on AN it follows that $\eta = \eta_1 + \cdots + \eta_q$ on A + B. By use of the formulas $L(\eta_i) = 0$, $(i = 1, \ldots, q)$, it is found that equation (3.2) holds. The criterion described in Theorem 3.2 is accordingly implied by the one given in Theorem 3.3. The converse is immediate. Theorem 3.3 now follows from Theorem 3.2.

Theorem 3.4. *The equation (2.1) holds for every admissible function $\eta(x)$ if and only if it holds with A, B replaced by A_0,*

B_0 for every admissible subregion A_0 of A whose boundary B_0 is also in A.

An admissible subregion A_0 of A is, of course, one whose boundary B_0 has the properties required for the boundary B of A. In this theorem the class of admissible functions associated with $A_0 + B_0$ is understood to be the class of admissible functions that is associated with the original set A + B. The result in Theorem 3.4 can be proved by use of Theorem 3.3. Let P be any point of A. There is a spherical neighborhood A_0 of P whose closure is in A. This region is obviously an admissible subregion of A. By virtue of the criterion of Theorem 3.4, equation (2.1) holds on subregion $A_0 + B_0$ for every admissible function $\eta(x)$ and, in particular, it holds if $\eta(x)$ vanishes on $A - A_0$. Thus the hypotheses of Theorem 3.3 are satisfied and application of this theorem completes the proof of Theorem 3.4.

Theorem 3.5. The results described in Theorems 3.1 to 3.4 hold when A is an interval represented by inequalities of the form $a_\alpha < x_\alpha < b_\alpha$, ($\alpha = 1, \ldots, m$).

This result follows from the more general result given in Theorem 5.2. A more elementary proof, however, is given below. The arguments for an interval A are like those given previously except in the proof of Theorem 3.2, where an additional argument is needed in order to show that equation (2.1) holds when the criterion described in Theorem 3.2 is satisfied. In order to prove Theorem 3.2 for an interval A let us suppose that the formula (2.1) holds for every admissible function $\eta(x)$ that is identically zero on a neighborhood of k ($k \leqq 2m$) of the faces of the interval A. We shall show that if k is greater than zero, then the formula (2.1) holds, also, for every admissible function having $\eta(x) \equiv 0$ on a neighborhood of K - 1 of the faces of A.

To this end, consider an admissible function $\eta(x)$ that is identically zero on a neighborhood of k - 1 faces of A. We can suppose that there is no neighborhood of the face $x_1 = a_1$ on which η is identically zero. For convenience, suppose $a_1 = 0$. Let $h(t)$ be the function used in the proof of Theorem 3.1 and set $H(x, r) = h(\frac{x_1}{r})$, where r is an arbitrary positive constant. Since $H = 0$ when $x_1 < r$, the variation $y = H\eta$ vanishes on a neighborhood of k of the faces of A and hence $L(y) = 0$, that is

$$(3.4) \quad \int_A (v_\alpha \pi_\alpha + w\eta)dA = -\int_A \eta v_\alpha H_{x_\alpha} dA + \int_{B_1} H\eta v_\alpha \lambda_\alpha dB,$$

where B_1 is the boundary of A with the face $x_1 = 0$ excluded. This face can be excluded, since $H = 0$ when $x_1 = 0$. As r tends to zero, the first and last integrals have respectively the limits

$$(3.5) \qquad \int_A (v_\alpha \pi_\alpha + w\eta)dA, \quad \int_{B_1} \eta v_\alpha \lambda_\alpha dB,$$

since $\lim_{r=0} H = 1$ on A + B except on the face $x_1 = 0$. The remaining integral can be written in the form

$$-\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} dx_2, \ldots, dx_m \int_0^{2r} \eta v_1 h'(\frac{x_1}{r})\frac{dx_1}{r} \qquad (2r < b_1)$$

since $H_{x_\alpha} = 0$ $(\alpha > 1)$, $H_{x_1} = h'(\frac{x_1}{r})/r$ and $H_{x_1} = 0$ when $x_1 > 2r$. Setting $x_1 = rt$ this integral becomes

$$-\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} dx_2, \ldots, dx_m \int_0^2 \eta v h'(t)dt.$$

Letting r tend to zero and using the relations $h(0) = 0$, $h(2) = 1$, we obtain as a limit the expression

$$(3.6) \quad -\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} v_1 dx_2, \ldots, dx_m = \int_{B_2} \eta v_\alpha \lambda_\alpha dB$$

where B_2 is the face $x_1 = 0$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the direction cosines $(-1, 0, \ldots, 0)$ of the outer normal of B_2 relative to A. In view of equation (3.4) and the limits (3.5) and (3.6), it follows that equation (2.1) holds for an admissible function $\eta(x)$ having $\eta(x) \equiv 0$ on a neighborhood of $k - 1$ of the faces of A. Letting k take successively the values $2m$, $2m - 1$, ..., 1 it is seen from the result just established that the criterion described in Theorem 3.2 implies that equation (2.1) holds for an interval A. This completes the proof of Theorem 3.5.

4. *Analogue of Green's Theorem.* If η is set identically equal to one in equation (2.1) this relation takes the form

$$\int_A w\, dA = \int_B v_\alpha \ell_\alpha\, dB. \tag{4.1}$$

This result suggests the following criterion for equation (2.1) to hold.

Theorem 4.1. *The equation* (2.1) *holds if and only if the formula*

$$\int_{A'} w\, dA = \int_{B'} v_\alpha \ell_\alpha\, dB \tag{4.2}$$

holds for every m-*dimensional cube* A' *whose closure is in* A *and whose faces are parallel to the coordinate hyperplanes.*

Let A' represent an m-dimensional cube on A at a distance $d > 0$ from the boundary B and whose faces are parallel to the coordinate hyperplanes. Each point of A is interior to a cube of this type. By Theorem 3.3, it will be sufficient to show that equation (2.1) holds for every admissible function $\eta(x)$ which vanishes on A - AA' when the criterion described in Theorem 4.1 is satisfied. We shall make use of the following integral means[*]

$$V_\alpha(x, r) \equiv \frac{1}{(2r)^m}\int_{A_r} v_\alpha(s)\, dA; \quad W(x, r) \equiv \frac{1}{(2r)^m}\int_{A_r} w(s)\, dA, \quad (0 < r < d), \tag{4.2}$$

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[*] For a discussion of integral means see, for example, Levin [XIV].

where $(s) \equiv (s_1, \ldots, s_m)$ and A_r is the region $x_\alpha - r \leqq s_\alpha \leqq x_\alpha + r$, $(\alpha = 1, \ldots, m)$. By hypothesis, we have

$$\int_{A'} w \, dA - \int_{B'} v_\alpha \chi_\alpha \, dB = 0.$$

When we divide each member of this equation by $(2r)^m$ and use relations (4.2), we find that $W - V_{\alpha x_\alpha} = 0$ on $A' + B'$.

If $\eta(x)$ is any admissible function, we have, by means of iterated integration, the relation

$$\int_{A'} (V_\alpha \pi_\alpha + W\eta) dA = \int_{A'} [\frac{\partial}{\partial x_\alpha}(V_\alpha \eta) + \eta(W - V_{\alpha x_\alpha})] dA = \int_{B'} \eta V_\alpha \chi_\alpha \, dB.$$

Thus, it is seen that

$$\int_{A'} (V_\alpha \pi_\alpha + W\eta) dA = \int_{B'} \eta V_\alpha \chi_\alpha \, dB.$$

Taking the limit as r tends to zero, the result is

$$\int_{A'} (v_\alpha \pi_\alpha + w\eta) dA = \int_{B'} \eta v_\alpha \chi_\alpha \, dB.$$

Thus, equation (2.1) holds for any admissible function $\eta(x)$ which vanishes on $A - AA'$. This result and the application of Theorem 3.3 complete the proof of the direct theorem. The proof of the converse follows immediately from Theorem 3.5.

Corollary 1. *The equation* (4.1) *holds for* A *when it holds for every* m-*dimensional cube whose closure is in* A *and whose faces are parallel to the coordinate hyperplanes*.

It is interesting to observe in passing, that the cubes $A' + B'$ used above are not admissible subregions in the sense described in Section 2, but Theorem 3.3 does not require that the neighborhoods N be admissible subregions of A.

As a consequence of Theorem 4.1, we have the following result.

Theorem 4.2. *Suppose that on each cube* $a_\alpha \leqq x_\alpha \leqq b_\alpha$ *in* A *the functions* v_α *are absolutely continuous in each variable* x_β *when the remaining variables are fixed, except on a set of* (m - 1)-*dimensional measure zero. Suppose, also, that* $v_{\alpha x_\alpha}$ *are integrable on* A. *Then equation* (2.1) *holds if and only if* $w = v_{\alpha x_\alpha}$ *almost everywhere on* A.

By Theorem 4.1, equation (2.1) holds if and only if equation (4.1) holds for all m-dimensional cubes A' whose faces are parallel to the coordinate hyperplanes. It remains to be shown that equation (4.1) holds for such cubes if and only if $w = v_{\alpha x_\alpha}$ almost everywhere on A.

By iterated integration it can be shown that

$$\int_{A'} v_{\alpha x_\alpha}\, dA = \int_{B'} v_\alpha \gamma_\alpha\, dB$$

where γ_α are direction cosines of the outer normal to B'. Equation (4.1) holds if $w = v_{\alpha x_\alpha}$. Therefore, we have the relation

$$\int_{A'} w\, dA = \int_{B'} v_\alpha \gamma_\alpha\, dB = \int_{A'} v_{\alpha x_\alpha}\, dA.$$

From this follows the relation

$$\int_{A'} (w - v_{\alpha x_\alpha})dA = 0$$

which holds for all cubes A' with sides parallel to the coordinate hyperplanes only if $w = v_{\alpha x_\alpha}$ almost everywhere on A. This completes the proof of Theorem 4.2.

We have as an immediate consequence of this result the following corollary which is discussed by Huke [XIII, pp. 55 ff.].

Corollary 1. *The equation*

$$\int_A w\eta\, dA = 0$$

holds for every admissible function $\eta(x)$ *with* $\eta \equiv 0$ *on* B, *if and only if* $w = 0$ *almost everywhere on* A.

This result follows from Theorems 3.2 and 4.2 with $v_\alpha = 0$. When $w = 0$ we have by these theorems the following further result.

Corollary 2. *If the functions* $v_\alpha(x)$ *are of class* c' *on* A, *the equation*

$$\int_A v_\alpha \pi_\alpha \, dA = 0 \tag{4.4}$$

holds for every admissible function η *having* $\eta = 0$ *on* B, *if and only if* $v_{\alpha x_\alpha} = 0$. *Moreover, if* $v_{\alpha x_\alpha} = 0$ *on* A *one has*

$$\int_{B'} v_\alpha \lambda_\alpha \, dB = 0 \tag{4.5}$$

where B' *is the boundary of an admissible subregion of* A.

Green's theorem may be obtained as an immediate consequence of Theorem 4.2. This constitutes a proof of the theorem for an m-dimensional region A, having a boundary B of the type described at the beginning of Section 2. The proof will be extended to hold for a region having a more general type of boundary in Section 5.

Theorem 4.3. (Green's theorem). *Under the hypotheses of Theorem* 4.2, *the equation* (4.1) *holds when* $w = v_{\alpha x_\alpha}$ *almost everywhere on* A.

By Theorem 4.2, equation (2.1) holds with $w = v_{\alpha x_\alpha}$. Setting $\eta = 1$ it is seen that equation (4.1) holds with $w = v_{\alpha x_\alpha}$, as was to be proved. This theorem is sometimes called Green's Lemma or Gauss's Theorem.

5. *Extension of the class of admissible regions*. In order to simplify the argument in the previous sections, we restricted ourselves to intervals and to regions having a boundary of class c' and non-singular. In this section, we shall consider a more general type of boundary as described in the following theorem.

Theorem 5.1. *The results described in Sections* 3 *and* 4 *are valid if the class of admissible regions* A *is extended to include regions whose boundary* B *can be subdivided into a finite number of non-singular* $(m - 1)$*-dimensional subspaces of class* c' *whose intersections form a set* E *having the following property: there exist positive numbers* q *and* s *with* $s < m - 1$ *such that, given a number* $r > 0$, *the quantity* q/r^s *is an upper bound of the number of points in a set* D, *on* E, *chosen so that the distance between any two points of* D *is at least* r.

Let r be a positive number and select a maximal set of points $P_1, \ldots, P_b$ on E, each pair of which are separated by a distance not less than r. Construct spheres of radius r with centers on $P_1, \ldots, P_b$. The spheres define a neighborhood N' of edge E. Consider, also, the neighborhood N of E which can be formed by constructing spheres of radius 2r and centers on $P_1, \ldots, P_b$. Observe that there is a number e independent of r and such that any sphere of radius 4r contains at most e of the points $P_1, \ldots, P_b$. Let $H_j(x, r) \equiv h(\frac{d_j(x)}{r})$, where $h(t)$ is the function defined in Section 3, and $d_j(x)$ is the distance of point (x) from the point P_j. Setting $H(x, r) \equiv H_1H_2 \ldots H_b$ it is seen that outside the neighborhood N of E, $H = 1$ and $H_{x_\alpha} = 0$, while inside the neighborhood N' of E, $H = 0$ and $H_{x_\alpha} = 0$. Moreover, since at most e of the functions H_j are different from 1, it follows that the absolute value of H_{x_α} is not greater than Ge/r, where G is an upper bound of $h'(t)$. Consider now an admissible function $\eta(x)$ and let $\zeta = H\eta$. Since $\zeta = 0$ in the neighborhood N' of E, it can be shown that equation (2.1) holds for ζ by an argument similar to that used in proof of Theorem 3.2, where the η used there is replaced by ζ. Thus we have

$$\int_A (v_\alpha \zeta_{x_\alpha} + w\zeta)\,dA = \int_B \zeta v_\alpha \ell_\alpha \,dB.$$

Replacing ζ by $H\eta$, we find that

$$\int_A H(v_\alpha \pi_\alpha + w\eta) + \int_A \eta v_\alpha H_{x_\alpha} \, dA = \int_B H\eta v_\alpha \lambda_\alpha \, dB.$$

In the limit as r tends to zero, this equation will become the equivalent of equation (2.1) if it can be shown that the second integral has zero as its limit. Let M be an upper bound of the values of ηv_α on A. Observe that the m-dimensional measure of AN cannot exceed $(q/r^s)cr^m$, where c is a constant. Since H_{x_α} is identically zero on A - AN, the absolute value of the second integral is not greater than

$$(Mm)(Ge/r)(q/r^s)cr^m = MmGeqc \cdot r^{m-s-1}$$

which tends to zero with r, since s < m - 1 and the remaining terms are constants. This completes the proof of Theorem 5.1.

Theorem 5.2. *The results described in Sections* 3 *and* 4 *hold if the class of admissible regions* A *is extended to include regions whose boundary* B *can be subdivided into a finite number of non-singular* (m - 1)-*dimensional subspaces of class* c' *having the following property: The boundary of each* (m - 1)-*dimensional subspace of* B *can be subdivided into a finite number of parts, each of which can be represented by a set of functions of the form*

$$(5.1) \qquad x_\alpha(t_1, \ldots, t_{m-2}), \qquad (t \text{ on } T; \ \alpha = 1, \ldots, m)$$

which are Lipschitzian on T, *the set* T *being bounded and closed.*

It will be sufficient to prove that the intersections of the (m - 1)-dimensional subspaces of B form a set E for which the hypotheses of Theorem 5.1 are satisfied. To do so we restrict ourselves to one piece of the set E given by equations of the form (5.1). This procedure is valid because if each piece of this type has the property described in Theorem 5.1, the whole set E has this property, since E is composed of a finite number of such pieces. Consider, therefore, one of the subdivisions E' of E given by equations (5.1) and let K be a positive number such that

$$\|x_\alpha(t) - x_\alpha(t')\| < K \|t - t'\| \tag{5.2}$$

where $\|x\|$ is the positive square root of $x_1^2 + \cdots + x_m^2$ and $\|t\|$ is the positive square root of $t_1^2 + \cdots + t_{m-2}^2$. Let r be a positive number and let $P_1, \ldots, P_b$ be a maximal set of points on the subset E' given by equations (5.1) and such that any two points are separated by a distance $d \geq r$. Let $T_1, \ldots, T_b$ be the points in T which correspond respectively to the points $P_1, \ldots, P_b$. By relation (5.2) the points $T_1, \ldots, T_b$ are at a distance $d \geq r/K$ apart. The number of points of this type will be shown not to exceed a quantity q/r^{m-2}, where q is a constant independent of r. To prove this we can suppose that T is an (m - 2)-dimensional cube of side c. We now choose the greatest integer $h < cK/r$. Hence $r < cK/h \equiv r'$. It is to be observed that $r \geq r'/2$ since $2h \geq cK/r$. If we divide T into h^{m-2} equal subcubes of side c/h, the number of points in a maximal set in one of these subcubes, at a distance $d \geq r'/2K = c/2h$ apart is exceeded by a number q' which is independent of h. The number of points in T in a maximal set at a distance $d \geq r/K \geq r'/2K$, therefore does not exceed

$$q'h^{m-2} = q'(cK/r')^{m-2} < q'(cK/r)^{m-2} = q/r^{m-2}$$

where $q = q'(cK)^{m-2}$ is a constant independent of r. This completes the proof of Theorem 5.2.

As a special case of this theorem for three-dimensional space we have the following result.

Theorem 5.3. *When region A is three-dimensional, the results of Sections* 3 *and* 4 *are valid for the case in which the* *boundary* B *is composed of non-singular surfaces of class* c' *which intersect in a finite number of rectifiable curves.*

This follows because the functions $x_\alpha(s)$ defining a rectifiable curve are Lipschitzian, when the parameter s is the arc length.

6. *Extension of the class of functions* v_α. In this section, we shall consider that the functions v_α are such that A + B can be sub-divided into a finite number of parts on each of which v_α are continuous and such that each part is a region of the type described in Section 5. The following result will be proved.

Theorem 6.1. *Equation* (2.1) *holds for every admissible function* $\eta(x)$ *if and only if the following two conditions hold:* (1) *at each point on an* (m - 1)-*dimensional manifold* R *of discontinuity of* v_α *the function* $v_\alpha \chi_\alpha$ *is continuous along the normal to* R, *whose direction cosines are represented by* χ_α; (2) *equation* (4.1) *holds for every* m-*dimensional cube* A' *which has its closure in* A, *its faces parallel to the coordinate hyperplanes, and contains no point of discontinuity of one of the functions* v_α.

First we shall prove that criteria (1) and (2) described in the theorem are sufficient to insure that equation (2.1) holds for every admissible function $\eta(x)$. Let $A_1, \ldots, A_c$ be the admissible subregions of A on which v_α are continuous and let $B_1, \ldots, B_c$ be their boundaries. By criterion (2) and Theorem 4.1, the formula (2.1) holds for each A_i and hence for their sum. We thus obtain

$$\int_A (v_\alpha \pi_\alpha + w\eta)\,dA = \int_B \eta v_\alpha \chi_\alpha \,dB + \sum_{i,j} \int_{R_{ij}} \eta(\chi_\alpha^{\,i} v_\alpha^{\,i} + \chi_\alpha^{\,j} v_\alpha^{\,j})\,dR, \tag{6.1)}$$

where the symbol R_{ij} represents the common boundary of the subregions A_i and A_j and the summation is taken over the A_i and A_j

which have a common boundary. The symbols λ^{i}_{α} and λ^{j}_{α} represent respectively the direction cosines of the outer normals to A_i and A_j. Since by criterion (1), the last integral has the value zero, we have that equation (2.1) holds for every admissible function $\eta(x)$.

On the other hand, if equation (2.1) holds for every admissible function $\eta(x)$ then it holds for each subregion A_i as described above, and hence equation (6.1) is satisfied. Since equation (2.1) holds for A + B, the last integral is zero for every admissible function $\eta(x)$, which is possible only if $\lambda_\alpha v_\alpha$ is continuous along the normal to R. Thus we have shown that criterion (1) is necessary.

To prove the necessity of criterion (2) consider a cube A' + B' as described in the theorem. Since equation (2.1) holds for every admissible function $\eta(x)$ on cube A' + B', take $\eta = 1$. Thus it is seen that equation (4.1) holds and that criterion (2) is necessary. Furthermore, by using Corollary 1 of Theorem 4.1, it can be shown that equation (4.1) must hold for each admissible subregion A_i on which the functions v_α are continuous.

7. *Applications*. In this section we shall prove a few theorems which are useful in mechanics and elsewhere. It is to be noted that the proofs hold for an m-dimensional region A with boundary B of the type described in Section 5.

Theorem 7.1. *Suppose the equations*

$$\int_{A_0} f(x)dA = \int_{B_0} u_{x_\alpha}(x)\lambda_\alpha \, dB, \tag{7.1}$$

$$\int_{A_0} g(x)dA = \int_{B_0} v_{x_\alpha}(x)\lambda_\alpha \, dB, \tag{7.2}$$

hold for every m-*dimensional cube* A_0 *whose closure is in* A *and whose faces are parallel to the coordinate hyperplanes; where* f(x) *and* $g(x)$ *are integrable in* A *and the functions* $u(x)$, $v(x)$ *are of class* c' *on* A + B. *Then the following equation holds*

$$(7.3)\qquad \int_A (ug - vf)\,dA = \int_B (uv_{x_\alpha} - vu_{x_\alpha})\lambda_\alpha\,dB.$$

By Theorem 4.1, the hypothesis concerning equation (7.1) implies that the equation

$$(7.4)\qquad \int_A (u_{x_\alpha}\pi_\alpha + f\eta)\,dA = \int_B \eta\, u_{x_\alpha}\lambda_\alpha\,dB$$

holds for any admissible function $\eta(x)$. Choosing $\eta = v$ in this equation we obtain

$$(7.5)\qquad \int_A (u_{x_\alpha}v_{x_\alpha} + vf)\,dA = \int_B vu_{x_\alpha}\lambda_\alpha\,dB.$$

By a similar method, we find the equation

$$(7.6)\qquad \int_A (v_{x_\alpha}u_{x_\alpha} + ug)\,dA = \int_B uv_{x_\alpha}\lambda_\alpha\,dB.$$

Now subtracting equation (7.5) from equation (7.6) the desired result is found.

As a consequence of Theorem 7.1, we have

Theorem 7.2. *Suppose that* $u(x)$ *and* $v(x)$ *are of class* c' *on* A + B *and that their second derivatives exist and are integrable on* A. *Then the following equation is satisfied*

$$(7.7)\qquad \int_A [u(\Delta v) - v(\Delta u)]\,dA = \int_B \left(u\frac{dv}{dn} - v\frac{du}{dn}\right)dB$$

where Δv *is the Laplacian* $\Delta v = v_{x_\alpha x_\alpha}$ *and* $\frac{dv}{dn}$ *is the outer normal derivative*.

By Theorem 4.2, it follows that equations (7.1) and (7.2) hold with $f = \Delta u$ and $g = \Delta v$, so that the desired result follows immediately from Theorem 7.1.

Theorem 7.3. *Suppose that* $v(x)$ *is of class* c' *on* A + B *and that its second derivatives exist and are integrable on* A.

Then if $u(x)$ *is an admissible function the following equation holds*

$$(7.7)\qquad \int_A (u_{x_\alpha} v_{x_\alpha} + u\,\Delta v)\,dA = \int_B u\frac{dv}{dn}.$$

By Theorem 4.2, equation (2.1) holds if we let $w = \Delta v$ and replace v_α by v_{x_α}. If η is then set equal to u the result is equation (7.7), which completes the proof of this theorem.

In order to find a further interesting result, consider a set of functions $R_{\alpha\gamma}(x)$, ($\alpha = 1, \ldots, m$; $\gamma = 1, \ldots, m - 1$) and the symbol $\left|\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}\right|$ which will represent the sum $\frac{\partial v_\alpha}{\partial x_\alpha}$, where v_α is the cofactor of l_α in the determinant $|l_\alpha R_{\alpha\gamma}|$. We are now in position to state the following theorem.

Theorem 7.4. *If the functions* $R_{\alpha\gamma}(x)$, ($\alpha = 1, \ldots, m$; $\gamma = 1, \ldots, m - 1$), *are of class* c' *on* A, *we have*

$$(7.8)\qquad \int_A \left|\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}\right| dA = \int_B \left| l_\alpha R_{\alpha\gamma}\right| dB\,.$$

This result may be proved immediately by use of Theorem 4.3. Let v_α be the cofactor of l_α in the determinant $|l_\alpha R_{\alpha\gamma}|$ and let $w = \left|\frac{\partial}{\partial x_\alpha} R_{\alpha\gamma}\right|$, then $w = \frac{\partial}{\partial x_\alpha} v_\alpha$. Application of Theorem 4.3 completes the proof.

8. *Further results.* We now restrict the admissible functions $\eta(x)$ to be non-negative on A + B and we shall consider the relations

$$(8.1)\qquad \int_A (v_\alpha \pi_\alpha + w\eta)\,dA \geqq \int_B \eta v_\alpha l_\alpha\,dB,$$

$$(8.2)\qquad \int_A w\,dA \geqq \int_B v_\alpha l_\alpha\,dB.$$

The theorems of Sections 3, 4 and 5 and their proofs hold if equations (2.1) and (4.1) are replaced respectively be relations (8.1) and (8.2). Results similar to these and their applications are discussed by Levin [XIV] for the two-dimensional case under less restrictive hypotheses on the functions v_α. The following theorem is formed by making the above mentioned changes in Theorems 3.2, 4.1 and 4.2. The boundary is considered to be of the type described in Section 5. The proofs are similar to those of the theorems just mentioned.

Theorem 8.1. *The inequality* (8.1) *holds for every non-negative admissible function* $\eta(x)$ *if and only if it holds for every non-negative admissible function* $\eta(x)$ *vanishing on a neighborhood of the boundary* B. *Moreover, the inequality* (8.1) *holds for these functions if and only if the inequality* (8.2) *is satisfied for every* m-*dimensional cube* A' *whose closure is in* A *and whose faces are parallel to the coordinate hyperplanes. If the functions* v_α *have the property described in Theorem* 4.2, *then the inequality* (8.1) *holds for every non-negative admissible function if and only if the relation* $w \geqq v_{\alpha x_\alpha}$ *holds almost everywhere on* A. *Finally, the above results hold if all inequalities are reversed or if all inequalities are replaced by equalities.*

9. *Formulation of the problem of the calculus of variations for multiple integrals.* In the remainder of this paper the symbol x will represent a point in m-dimensional space X, as before, and y will represent a point in n-dimensional space Y. The symbols A and B will have the same significance as before, the boundary B being of the type described in Section 5. The partial derivatives $\frac{\partial y_i}{\partial x_\alpha}$ will be represented by p_i^α. Here and in the remaining sections the index i will range from 1 to n and α from 1 to m. Let R be a region of XYP space. A function y(x) will be said to be of class D' on A + B in case it is such that A + B can be subdivided into a finite number of admissible subregions on each of which the function y(x) is of class c'. An *admissible manifold* will be one having all of its elements

interior to R and with defining functions $y_i(x)$ of class D'.

Our problem is then to find among all admissible manifolds

$$y = y(x), \quad (x \text{ in } A)$$

having a common boundary L, one which minimizes the m-tuple integral

$$I = \int_A f(x, y, p)\,dA \tag{9.1}$$

where f is of class c'' in the region R.

10. *Necessary conditions associated with the first variation*. Let E be an admissible manifold defined by $y_i(x)$ and consider a one-parameter family of admissible manifolds

$$y = y(x) + \epsilon\eta(x)$$

where $\eta(x)$ is an admissible manifold vanishing on B. If we replace $y(x)$ by $y(x) + \epsilon\eta(x)$ in the integral (8.1), the result is

$$I(\epsilon) = \int_A f[x, y(x) + \epsilon\eta(x), \overline{p}]\,dA$$

where $\overline{p}$ represents the partial derivatives $\frac{\partial}{\partial x_\alpha}[y_i(x) + \epsilon\eta(x)]$. If E is a minimizing manifold, then for ϵ sufficiently small $y(x) + \epsilon\eta(x)$ is an admissible manifold and $I(\epsilon)$ must have a minimum for $\epsilon = 0$. Differentiation of $I(\epsilon)$ with respect to ϵ yields

$$I'(\epsilon) = \int_A (f_{y_i}\eta_i + f_{p_i^\alpha}\pi_i^\alpha)\,dA \tag{10.1}$$

where the symbol π_i^α represents $\frac{\partial\eta_i}{\partial x_\alpha}$. Evaluating $I'(\epsilon)$ for $\epsilon = 0$, we have

$$I'(0) = \int_A (f_{y_i}\eta_i + f_{p_i^\alpha}\pi_i^\alpha)\,dA \tag{10.2}$$

which is called the *first variation* of the integral I along manifold E. A necessary condition for E to be a minimizing manifold is that $I'(0) = 0$ for every admissible variation $\eta(x)$ vanishing on B. Theorem 6.1 can be used to establish the following result.

Theorem 10.1. *When* $y(x)$ *defines a minimizing manifold* E *of class* D', *the equation*

$$(10.3) \qquad \int_A (f_{y_i}\eta_i + f_{p_i^\alpha}\pi_i^\alpha)\,dA = \int_B \eta_i f_{p_i^\alpha}\lambda_\alpha\,dB$$

holds for every admissible variation $\eta(x)$. *Moreover, equation* (10.3) *holds on* E *for every admissible variation* $\eta(x)$ *if and only if the following two conditions are satisfied:* (1) *at each point on an* $(m - 1)$*-dimensional manifold* R *of discontinuity of* $f_{p_i^\alpha}$ *the function* $f_{p_i^\alpha}\lambda_\alpha$ *is continuous along the normal to* R, *whose direction cosines are represented by* λ_α; (2) *the equations*

$$(10.4) \qquad \int_{A_0} f_{y_i}\,dA = \int_{B_0} f_{p_i^\alpha}\lambda_\alpha\,dB$$

hold for every m*-dimensional cube* A_0 *which has its closure in* A, *its faces parallel to the coordinate hyperplanes, and contains no point of discontinuity of one of the functions* $f_{p_i^\alpha}$.

It is to be noted that by Corollary 1 of Theorem 4.1, criterion (2) implies that equations (10.4) are satisfied for every subregion $A_0 + B_0$ of A on which the functions $y(x)$ are of class c'. These are known as the Haar-Coral [VI] equations.

By use of Theorem 4.2, we can establish the following result.

Theorem 10.2. (Euler equations). *On every portion of a minimizing manifold* E *where the functions* $y_i(x)$ *are of class* c", *we have*

$$(10.5) \qquad f_{y_i} - \frac{\partial}{\partial x_\alpha} f_{p_i^\alpha} \equiv 0.$$

As a further result we have [cf. XIV]

Theorem 10.3. *If an admissible manifold* E *defined by* *functions* $y_i(x)$ *of class* c' *affords a minimum to* I *in the class of admissible manifolds defined by functions* $Y_i(x)$ *for which* $Y_i(x) \geqq y_i(x)$ *on* A, *then the inequality*

$$\int_A (f_{y_i}\eta_i + f_{p_i^\alpha} \pi_i^\alpha)dA \geqq \int_B \eta_i f_{p_i^\alpha} \lambda_\alpha \, dB \tag{10.6}$$

holds on E *for all non-negative admissible variations* η_i. *The inequality* (10.6) *holds on* E *for these variations if and only if one has*

$$\int_{A'} f_{y_i} \, dA \geqq \int_{B'} f_{p_i^\alpha} \lambda_\alpha \, dB \tag{10.7}$$

on every cube A' *whose closure is in* A *and whose faces are parallel to the coordinate hyperplanes. If* E *minimizes* I *in the class of admissible manifolds defined by functions* $Y_i(x)$ *for which* $Y_i(x) \leqq y_i(x)$ *on* A, *the inequalities in* (10.6) *and* (10.7) *are reversed.*

This result follows from Theorem 8.1 since if E minimizes I in class of functions $Y_i(x)$ for which $Y_i(x) \geqq y_i(x)$ on A, then the first variation of I for $Y_i(x)$ satisfies the relation $I' \geqq 0$ for non-negative admissible variations $\eta_i(x)$ vanishing on B. If, on the other hand, E minimizes I in the class of admissible manifolds defined by functions $Y_i(x)$ for which $Y_i(x) \leqq y_i(x)$ one has $I'(0) \leqq 0$ for non-negative admissible variations vanishing on B, so that the inequalities (10.6) and (10.7) are reversed.

11. *Further necessary conditions.* In this section we find further necessary conditions by transforming the problem into parametric form. We shall prove the following result.

Theorem 11.1. *Let* E *be a minimizing manifold defined by functions* y(x). *On every admissible subregion* $A_0 + B_0$ *of* A *on which the functions* y(x) *are of class* c', *the following equations hold*

$$(11.1)\quad \int_{A_0} f_{x_\alpha}\, dA = \int_{B_0} (f\delta_{\alpha\beta} - p_1^\alpha f_{p_1^\beta})\gamma_\beta\, dB, \quad (\beta = 1, \ldots, m).$$

Moreover, at each point P on a manifold R of discontinuity of the functions $f_{p_1^\beta}$ *the function*

$$(11.2)\qquad (f\delta_{\alpha\beta} - p_1^\alpha f_{p_1^\beta})\gamma_\beta$$

is continuous along the normal to R, where γ_β *represent the direction cosines of the normal.*

In order to prove this result, we apply the transformation

$$(11.3)\qquad x_\alpha = x_\alpha(t_1, \ldots, t_m)$$

to the integral I. Then we have $y(x) = y(x(t)) = Y(t)$ and after differentiating, the result is

$$(11.4)\qquad r_1^\beta = p_1^\alpha x_{\alpha t_\beta}$$

where the notation r_1^β represents Y_{1t_β}. After solving for p_1^α, the result is

$$(11.5)\qquad p_1^\alpha = \frac{\Lambda_{1\alpha}}{\Delta}$$

where $\Delta \equiv |x_{\alpha t_\beta}|$ and $\Lambda_{1\alpha}$ is the determinant obtained from Δ by replacing the α-th row by r_1^β. These notations were used by Powell [XX, pp. 53-4]. The integral I then takes the form

$$(11.6)\qquad \int_T F(x, y, q, r)\,dT$$

where $F(x, y, q, r) = f[x(t), Y(t), \frac{\Lambda}{\Delta}]\Delta$, $q_\alpha^\beta \equiv (x_{\alpha t_\beta})$, and T is the region in t-space corresponding to region A_0 by transformation (11.3). This is a problem of the same type as the original except that now the independent variable is t and we have m + n dependent variables x(t) and Y(t). By application of Theorem 10.1 to this problem, we have

$$(11.7)\qquad \int_T F_{x_\alpha}\, dT = \int_S F_{q^\beta_\alpha} L_\beta\, dS$$

where S is the boundary of T, and L_β are direction cosines of the outer normal to S.

The partial derivatives $F_{q^\beta_\alpha}$ were calculated by Powell [XX, pp. 54-5] and were found to be

$$(11.8)\qquad \begin{aligned} F_{q^\alpha_\alpha} &= f - p^\alpha_i f_{p^\alpha_i} && (\alpha \text{ not summed}) \\ F_{q^\beta_\alpha} &= - p^\alpha_i f_{p^\beta_i} && (\alpha \neq \beta). \end{aligned}$$

These results may be stated in the form

$$(11.9)\qquad F_{q^\beta_\alpha} = f \delta_{\alpha\beta} - p^\alpha_i f_{p^\beta_i} .$$

Thus, it is seen that when the transformation $t_\alpha = x_\alpha$ is applied to integral (11.6) the result is

$$(11.10)\qquad \int_{A_0} f_{x_\alpha}\, dA = \int_{B_0} (f \delta_{\alpha\beta} - p^\alpha_i f_{p^\beta_i})\gamma_\beta\, dB.$$

This completes the proof of Theorem 11.1.

We have, also, the further result

Theorem 11.2. *On every portion of a manifold* E *where the functions* $y_i(x)$ *are of class* c", *the following identity holds.*

$$(11.11)\qquad f_{x_\alpha} - \frac{\partial}{\partial x_\beta}(f\delta_{\alpha\beta} - p^\alpha_i f_{p^\beta_i}) + p^\alpha_i \left(f_{y_i} - \frac{\partial}{\partial x_\beta} f_{p^\beta_i}\right) \equiv 0.$$

From this result it follows that equations (11.1) are a consequence of equations (10.4) or (10.5) for a minimizing manifold E of class c".

12. *Necessary conditions associated with the second variation.* Throughout this section we shall consider a minimizing manifold E whose defining functions $y_i(x)$ are of class c'. If we differentiate equation (10.1) with respect to ϵ and evaluate at $\epsilon = 0$ we have the *second variation*

$$I''(0) = \int_A 2\Omega(x, \eta, \pi)\,dA \tag{12.1}$$

of the integral I along manifold E; where $\pi = (\frac{\partial \eta_i}{\partial x_\alpha})$ and $2\,\Omega$ denotes the quadratic form $f_{y_i y_k}\eta_i\eta_k + 2f_{y_i p_i^\beta}\eta_i\pi_k^\beta + f_{p_i^\alpha p_k^\beta}\pi_i^\alpha\pi_k^\beta$, $(k = 1, \cdots, n;\ \beta = 1, \cdots, m)$.

For any minimizing manifold $y(x)$ the relations $I'(0) = 0$ and $I''(0) \geqq 0$ must hold. By Euler's theorem on homogeneous functions, we have

$$2\,\Omega = \Omega_{\pi_i^\alpha}\pi_i^\alpha + \Omega_{\eta_i}\eta_i. \tag{12.2}$$

By *Legendre's condition** in the stronger form is meant that along a minimizing manifold the quadratic form

$$f_{p_i^\alpha p_i^\beta} w_\alpha^i w_\beta^k \tag{12.3}$$

is positive for all matrices w_α^i of rank one.

We are now in a position to prove the following result.

Theorem 12.1. (Analogue of Jacobi's condition). *Let* $y = y(x)$ *be a minimizing manifold* E *of class* c' *on which Legendre's condition in the stronger form is satisfied. Furthermore, let* A_0 *be an admissible subregion of* A *with boundary* B_0. *Then there can be no solution* $\eta_i = u_i(x)$ *of the* n *equations*

$$\int_{A'}\Omega_{\eta_i}\,dA = \int_{B'}\Omega_{\pi_i^\alpha}\chi_\alpha\,dB, \qquad (i = 1, \ldots, n) \tag{12.4}$$

for all subregions A' *of* A_0 *and such that the functions* $u_i(x)$ *are of class* c' *on* $A_0 + B_0$, *vanish on* B_0, *and have first partial derivatives not all identically zero at the points of* B_0 *in* A.

In order to prove this theorem, let us suppose that the $u_i(x)$ satisfy the n equations (12.4), being of class c' on $A_0 + B_0$, vanishing on B_0 and such that their first partial derivatives are not all zero at a point P of B_0 in A. A portion E of B_0 in a neighborhood of P can be represented by the equations

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*For proofs of this necessary condition see the papers V, VII, IX and X at the end of this paper.

$$(12.5) \qquad x_\alpha = g_\alpha(t_1, \ldots, t_{m-1}), \qquad \sum \chi_\alpha^2 \neq 0,$$

where χ_α represents the direction cosines of the normal to B_0, and the functions g are of class c'.

Define the variation

$$(12.6) \qquad \begin{aligned} \eta_i(x) &\equiv u_i(x) \quad \text{on } A_0 + B_0 \\ &\equiv 0 \qquad\quad \text{outside } A_0 + B_0. \end{aligned}$$

Making use of relation (12.2) we find that the second variation for $\eta(x)$ is

$$(12.7) \qquad I''(0) = \int_A 2\Omega(x, \eta, \pi)\,dA = \int_A (\Omega_{\pi_i^\alpha} \pi_i^\alpha + \Omega_{\eta_i}\eta_i)\,dA.$$

Let us now consider the accessory problem of minimizing the integral (12.7) in the class of admissible manifolds $\eta(x)$. We have, by Theorems 6.1 and 3.2 that

$$(12.8) \qquad I''(0) = \int_{A_0} (\Omega_{\pi_i^\alpha} \pi_i^\alpha + \Omega_{\eta_i}\eta_i)\,dA = 0$$

since η vanishes on B. Thus I"(0) takes its minimum value for the variation $\eta(x)$. Therefore, $\eta_i(x)$ must satisfy the edge conditions near P, since these functions constitute a solution of class D' for the problem of minimizing I"(0). Thus, we have

$$\chi_\alpha \Omega_{\pi_i^\alpha} \Big|^1 = \chi_\alpha \Omega_{\pi_i^\alpha} \Big|^2$$

where the numbers 1 and 2 indicate that the limiting values are approached along the normal, first from one side of the edge E and then from the other side. The right members of these equations vanish, leaving

$$(12.9) \qquad \chi_\alpha \Omega_{\eta_i^\alpha}\,(x,\ u,\ w)\Big|^1 = 0$$

on E, where $w \equiv \frac{\partial u_k}{\partial x_\alpha}$.

In addition to equations (12.9) we have at P the equations

$$(12.10) \qquad du_i = \frac{\partial u_i}{\partial x_\alpha} dx_\alpha = \frac{\partial u_i}{\partial x_\alpha}\frac{\partial g_\alpha}{\partial t_\gamma} dt_\gamma = 0, \qquad (\gamma = 1,\ \ldots,\ m - 1).$$

Since the differentials dt_γ are arbitrary, equation (12.10) yields the equations

$$(12.11) \qquad \frac{\partial u_i}{\partial x_\alpha}\frac{\partial g_\alpha}{\partial t_\gamma} = 0, \quad \text{or} \quad w_i^\alpha\, g_\alpha^\gamma = 0.$$

Leaving i fixed and solving for w_i^α gives

$$w_i^\alpha = \lambda_i \chi_\alpha.$$

Multiply through equations (12.9) by λ_i and the result is

$$\begin{aligned} \lambda_i \chi_\alpha \Omega_{\eta_i^\alpha} &= \Omega_{\eta_i^\alpha}\,(\lambda_i \chi_\alpha) = \Omega_{\eta_i^\alpha}\, w_i^\alpha = \eta_k^\beta\, f_{p_i^\alpha\, p_k^\beta}\, w_i^\alpha \\ &= u_k^\beta\, f_{p_i^\alpha\, p_i^\beta} \lambda_i \chi_\alpha = f_{p_i^\alpha\, p_k^\beta} \lambda_i \chi_\alpha \lambda_k \chi_\beta = 0. \end{aligned}$$

This contradicts the Legendre condition, since the matrix w_i^α is of rank one. Then the admissible variation (12.6) cannot give I"(0) a minimum, which contradicts the hypothesis that y(x) is a minimizing manifold. This completes the proof of Theorem 12.1.

As a final result we have the following generalization of Theorem 7.2.

Theorem 12.2. *If* ξ_i *and* η_i *are admissible variations of class* c" *then the equations*

$$(12.12) \qquad \int_{A_0} [\xi_i J_i(\eta) - \eta_i J(\xi)\,|\,dA = \int_{B_0} (\eta_i \Omega_{k_i^\alpha} - \xi_i \Omega_{\pi_i^\alpha})\chi_\alpha\, dB$$

hold for every admissible subregion A_0 *of* A, *where* $J_i(\eta)$ *are the accessory equations*

$$(12.13) \qquad J_i(\eta) = \Omega_{\eta_i} - \frac{\partial}{\partial x_\alpha}\Omega_{\pi_i^\alpha}.$$

The symbol k_i^α *represents* ξ_{ix_α}, *the arguments in* $\Omega_{k_i^\alpha}$ *and* $\Omega_{\pi_i^\alpha}$ *are* (x, ξ, k) *and* (x, η, π), *respectively, and* ℓ_α *are the direction cosines of the outer normal to* B_0.

In order to prove this result, let us form the identity

$$\int_{A_0} \xi_i J_i(\eta)\,dA = \int_{A_0} (\xi_i \Omega_{\eta_i} + \xi_i \Omega_{\pi_i^\alpha})\,dA - \int_{A_0} \frac{\partial}{\partial x_\alpha}(\xi_i \Omega_{\pi_i^\alpha})\,dA \tag{12.13}$$

which, by Green's theorem, takes the form

$$\int_{A_0} \xi_i J(\eta)\,dA = \int_{A_0} (\xi_i \Omega_{\eta_i} + \xi_i \Omega_{\pi_i^\alpha})\,dA - \int_{B_0} \xi_i \Omega_{\pi_i^\alpha} \ell_\alpha\,dB. \tag{12.14}$$

Similarly, we can form the equation

$$\int_{A_0} \eta_i J_i(\xi)\,dA = \int_{A_0} (\eta_i \Omega_{\xi_i} + \eta_i \Omega_{k_i^\alpha})\,dA - \int_{B_0} \eta_i \Omega_{k_i^\alpha} \ell_\alpha\,dB. \tag{12.15}$$

Subtracting equation (12.15) from (12.14) and making use of relation (12.2), the result is equation (12.12). This completes the proof of Theorem 12.2.

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Most of the papers in the preceding chronological and alphabetical lists appear in the list below, but not all of them. A classification by subject matter is difficult to make, and is partly a matter of opinion. The one here given should not be regarded as exhaustive or infallible. It has been useful to the editors who hope that it may prove useful to others also.

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The conditions of Weierstrass and Legendre
The second variation and Jacobi's condition
Transformations of the second variation
Fields, invariant integrals, sufficiency theorems
Parametric problems
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